

Engineering Economy Outline

IE 305-Part 1

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The very basic notion that governs all of simple engineering, business, and personal finance is that if

i = an applicable per-period interest rate

an amount P in the present is economically equivalent to an amount

$$P(1 + i)$$

one period later. Of course, if that same interest rate is applicable through N consecutive periods, the amount P in the present is equivalent to an amount $P(1 + i)^N$ at the end of those N periods. If one uses the notation F for the single equivalent "future" value, one might write

$$F = P(1 + i)^N \quad (1)$$

Solving for P

The (N -period-constant-interest-rate) relationship (1) involves only four variables and can be solved for any of them to produce:

1. the **compound amount** formula

$$F = P(1 + i)^N$$

with corresponding EXCELTM function $FV(i, N, 0, P)$. The multiplier of P here is often denoted as $(F/P, i, N)$.

2. the **present worth** formula

$$P = F(1 + i)^{-N}$$

with corresponding EXCELTM function $PV(i, N, 0, F)$. The multiplier of F here is often denoted as $(P/F, i, N)$.

Solving for i and N

3. the interest rate that produces a ratio F/P in N periods

$$i = \left(\frac{F}{P}\right)^{1/N} - 1 \quad .$$

4. the number of periods that produce a ratio F/P under interest rate i

$$N = \frac{\ln\left(\frac{F}{P}\right)}{\ln(1+i)} \quad .$$

In regard to the last of these, there is the famous "Rule of 72" that says the number of periods required to double an initial value at interest rate i is approximately $72/(100i)$. This is simply a manifestation of the mathematical fact that

$$\frac{\ln(2)}{\ln(1+i)} \approx \frac{72}{100i}$$

Series of Cash Flows and Interest Rates

A fundamental activity of engineering economics is the analysis of the value of a series of cash flows (we'll call those that are positive from the analysis perspective "inflows" and those that are negative "outflows") at discrete times $t = 0, 1, 2, \dots, N$. With

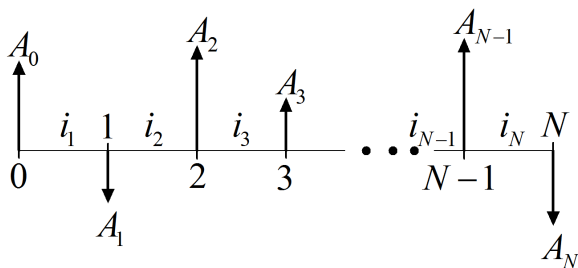
$A_n =$ the (signed) cash flow at time n

and

$i_n =$ the per-period interest rate operating in the n th period

these can be represented as on the diagram on the next slide. The A_n are portrayed as vectors located at points n in time. (The interest rates relevant between consecutive discrete time points have placed on the diagram between those points.)

Series of Cash Flows and Interest Rates



Based on the $N = 1$ version of the basic equivalence (1) and the notion that multiple (signed) cash flows at a given time point can be added to give a single equivalent one, it is possible to represent cash and interest series by **a single economically equivalent cash flow at any chosen time point, n** . The $n = 0$ case might be called the "present value" and the $n = N$ case might be called the "future value."

Analysis of General Cash Flow and Interest Rate Series

Operationally, in developing economic equivalences among cash flow series for a given series of interest rates, one may

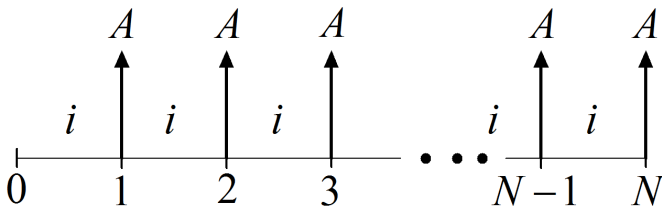
- (if convenient) decompose a single series one-time-point-at-a-time into two series that sum (one-time-point-at-a-time) to the original one, and compute separately and add (one-time-point-at-a-time) economically equivalent series for the two components,
- "move" a cash flow at a given point one period to the right by *multiplying by* $1 + i$ plus the interest rate operating in the period between,
- "move" a cash flow at a given point one period to the left by *dividing by* $1 + i$ plus the interest rate operating in the period between, and
- add (signed) cash flows at any time point to reduce them to a single value

and thereby maintain economic equivalence.

Constant Interest Rate Formulas-Equal Payment (Uniform) Series

Special forms for the cash flows $A_0, A_1, A_2, \dots, A_{N-1}, A_N$ together with a constant interest rate assumption lead to useful special formulas enabling efficient development of the details of standard financial contracts. We provide some of these, beginning with the most important case of "equal payment series."

So suppose that $A_0 = 0$ and $A_1 = A_2 = \dots = A_N = A$ (and interest rates are constant at i). This, of course, can be represented as below.



Equal Payment Series-Time N Economic Value Formulas

It is a consequence of some "simple" algebra that a time N economically equivalent value for an equal payment series is the **compound amount**

$$F = A \left[\frac{(1+i)^N - 1}{i} \right]$$

with corresponding EXCELTM function $FV(i, N, A)$. The multiplier of A here is often denoted as $(F/A, i, N)$ (and called the equal payments **compound amount factor**).

Solving the above equation for A produces

$$A = F \left[\frac{i}{(1+i)^N - 1} \right]$$

with corresponding EXCELTM function $PMT(i, N, 0, F)$. The multiplier of F here is often denoted as $(A/F, i, N)$ (and called the equal payments **sinking fund factor**).

Equal Payment Series-Time N Formulas

Two less common calculations based on the basic equation used on slide 9 concern solving that equation for N or i .

Simple algebra gives

$$N = \frac{\ln\left(\left(\frac{F}{A}\right) i + 1\right)}{\ln(i + 1)}$$

as the number of compounding periods required to produce a given ratio F/A (of time N economic value to payment amount) at per-period interest rate i .

Solution of

$$\frac{F}{A} = \left[\frac{(1 + i)^N - 1}{i} \right]$$

for interest rate i given inputs F/A and N is a numerical problem, presumably doable using GoalSeek in EXCELTM.

Equal Payment Series-Time 0 Formulas

Setting $F = P(1+i)^N$ in the basic formula on slide 9 and solving for P produces the time $n = 0$ formula for the **present worth** of an equal payment series

$$P = A \left[\frac{(1+i)^N - 1}{i(1+i)^N} \right]$$

with corresponding EXCELTM function $PV(i, N, A)$. The multiplier of A here is often denoted as $(P/A, i, N)$ (and called the equal payments **present worth factor**).

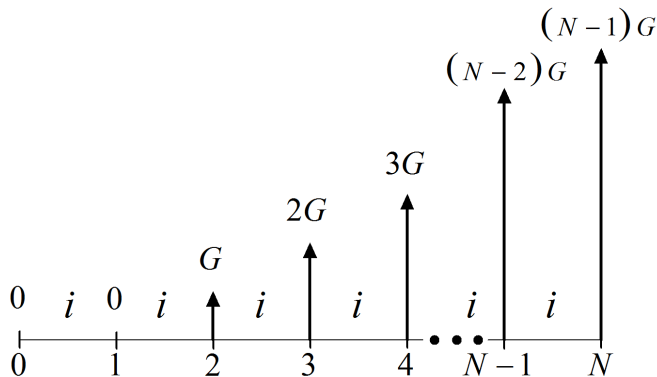
Solving the above equation for A gives the time $n = 0$ formula

$$A = P \left[\frac{i(1+i)^N}{(1+i)^N - 1} \right]$$

with corresponding EXCELTM function $PMT(i, N, A)$. The multiplier of P here is often denoted as $(A/P, i, N)$ (and called the equal payments **annuity factor** or **capital recovery factor**).

Constant Interest Rate Formulas-Linear Gradient Series

We next consider a particular form of linearly increasing cash flow series. Namely, we consider equivalent economic values (under constant interest rate i) for a series with $A_0 = 0, A_1 = 0, A_2 = G, A_3 = 2G, A_4 = 3G, \dots, A_{N-1} = (N-2)G, A_N = (N-1)G$. (For $n \geq 1$, we're considering $A_n = (n-1)G$, a situation illustrated below.)



Linear Gradient Series Formulas

It is a consequence of some "simple" algebra that a time 0 economically equivalent value for a linear gradient series (under constant interest rate i) is

$$P = G \left[\frac{(1+i)^N - iN - 1}{i^2 (1+i)^N} \right]$$

The multiplier of G here might be denoted as $(P/G, i, N)$ (and called the linear gradient series **present worth factor**).

Of course one could write $P = (1+i)^{-N} F$ above and then easily derive the time N economically equivalent value for the linear gradient series as

$$F = G \left[\frac{(1+i)^N - iN - 1}{i^2} \right]$$

And the above equations are easily solved for G in terms of i , N and P or F .

Linear Gradient Series Formulas

Various other (apparently less interesting) problems could be stated for linear series.

- Something like GoalSeek in EXCEL™ might be used to solve the equations on slide 13 for i or N .
- The text considers the problem of finding an equal payment amount A producing a series with the same single time point economic values linear gradient series with gradient G . This is simply accomplished using the gradient series present worth factor and the equal payment series capital recovery factor to produce a factor

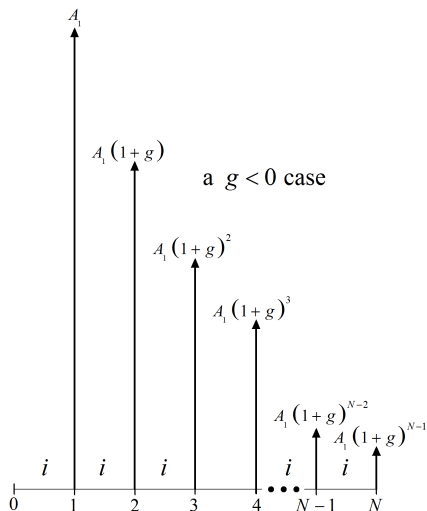
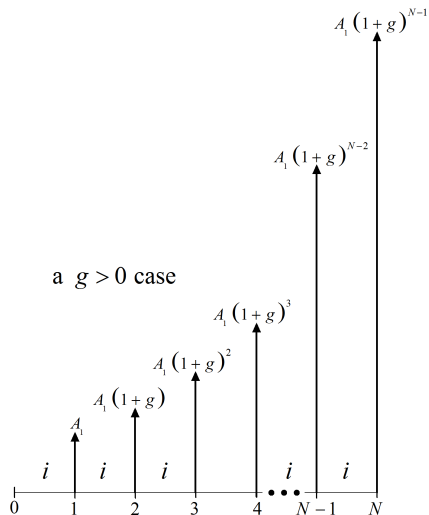
$$\begin{aligned}(A/G, i, N) &\equiv (P/G, i, N) \cdot (A/P, i, N) \\ &= \left[\frac{(1+i)^N - iN - 1}{i \left((1+i)^N - 1 \right)} \right]\end{aligned}$$

Constant Interest Rate Formulas-Geometric Gradient Series

We next consider a particular form of geometrically changing cash flow series. This might be relevant where some expense is expected to increase by some percentage factor yearly. For some fixed constant g , we consider equivalent economic values (under constant interest rate i) for a series with $A_0 = 0$, A_1 , $A_2 = A_1(1 + g)$, $A_3 = A_1(1 + g)^2$, $A_4 = A_1(1 + g)^3$, ..., $A_{N-1} = A_1(1 + g)^{N-2}$, $A_N = A_1(1 + g)^{N-1}$. (For $n \geq 1$, we're considering $A_n = A_1(1 + g)^{n-1}$.)

Such a series is 0 at time $n = 0$, begins at A_1 at time $n = 1$ and then increases or decreases geometrically depending upon the sign of g as illustrated next.

Geometric Gradient Series



Geometric Gradient Series Formulas

It is a consequence of some "simple" algebra that a time 0 economically equivalent value for a geometric gradient series (under constant interest rate i) is

$$P = A_1 \times \begin{cases} \frac{1}{i-g} \left[1 - \left(\frac{1+g}{1+i} \right)^N \right] & \text{if } i \neq g \\ \frac{N}{1+i} & \text{if } i = g \end{cases}$$

The multiplier of A_1 here might be denoted as $(P/A_1, g, i, N)$ (and called the geometric gradient series **present worth factor**).

Of course, it's possible to replace P above with $F(1+i)^{-N}$ and solve for a time N economically equivalent value of a geometric gradient series. Other problems could be phrased (in terms of solving for i, g , or N in terms of P/A_1 or F/A_1 and the others of those quantities). But those seem far less interesting than simple computation of present or future equivalent values.

Nominal and Effective Interest Rates

Business/commerce uses a variety of schedules for stating and charging interest. The generic "period" we've used thus far could be a day, a week, a month, a year, etc. It's important to understand exactly how interest rates are stated and how they are applied to a series of cash flows and corresponding economic values of that series are then computed. The difficulties we face are that a commercial interest rate may not be stated in terms of the compounding period and that cash flows may not be on the same schedule/at the same frequency at which interest is compounded. Somehow these must be reconciled in a coherent way.

Effective Rates Where Compounding Periods are a Fraction of the Period Over Which a Rate is Stated

It is quite common to hear interest rates stated in terms of "Annual Percentage Rate" (APR) in contexts where compounding may be done quarterly, monthly, weekly, or daily. In such cases there are some number (4, 12, 52, or 365 e.g.) of compounding periods in the larger period for which the nominal rate is stated. What do we make of this?

The standard understanding of this language (e.g. 7.0% APR compounded monthly) is the following. For

r = a nominal interest rate stated to cover M compounding periods

a per-compounding-period interest rate of

$$\frac{r}{M}$$

will be applied/compounded at each of the M periods.

Effective Interest Rates

If the rate r/M is compounded M times, an amount P grows to an amount

$$P \left(1 + \frac{r}{M}\right)^M$$

over the period for which r is the nominal rate. That means that over the M periods the growth is the same as that which would have been provided by a *single* compounding with the rate

$$i_{\text{effective}} = \left(1 + \frac{r}{M}\right)^M - 1$$

In this way a 4% APR is an effective annual rate of 4.06% if compounding is quarterly ($M = 4$) and 4.07% if compounding is monthly ($M = 12$) and 4.08% if compounding is daily ($M = 365$).

Continuous Compounding

The calculus fact that

$$\lim_{M \rightarrow \infty} \left(1 + \frac{r}{M}\right)^M = e^r$$

means that the effective interest rate for a nominal rate of r for extremely frequent compounding is essentially $e^r - 1$. In fact, one can consider **continuous compounding** with nominal interest rate r for some specified period. An amount P subject to such a scheme for k times that period (k can be any positive number) grows to

$$e^{kr} P$$

Analysis of Cash Flows on a Different Schedule Than Compounding

If there are M compounding periods in the period over which a nominal interest rate r is stated, one uses the interest rate r/M per compounding period. If the compounding period is k times the length of the period over which a rate r is stated, one uses the compounding-period rate $(1 + r)^k - 1$.

For analysis purposes, cash flows must be the same schedule as interest compounding. So we consider what to do if there are C compounding periods per cash flow period and the two cases:

1. the possibility that C is a positive integer (so compounding is more frequent than cash flows), and
2. the possibility that $1/C$ is a positive integer (so cash flows are more frequent than compounding).

Compounding More Frequent than Cash Flows

In the case where the number of compounding periods per cash flow period (C) is a positive integer, one can simply do economic analysis *on the cash flow time scale* provided the right interest rate is used. For $i_{\text{compounding}}$ the relevant interest rate for the compounding period, the per-cash-flow-period interest rate is

$$i_{\text{cashflow}} = (1 + i_{\text{compounding}})^C - 1$$

Cash Flows More Frequent than Compounding

Where the number of compounding periods per cash flow period is $(1/C)$ for a positive integer C (there are C cash flow periods per compounding period) there are two standard ways of proceeding. One may

1. find a per-cash-flow-period interest rate with corresponding per-compounding-period effective rate that is the stated one, and use that rate compounding on the cash flow schedule (that is, one might set

$$i_{\text{compounding}} = (1 + i_{\text{cashflow}})^{1/C} - 1$$

and solve to produce

$$i_{\text{cashflow}} = (1 + i_{\text{compounding}})^C - 1$$

and operate on the cash flow schedule), or

2. move all cash flows between compounding points to the following compounding point, effectively assuming that they have no interest consequences, having not been considered for a full compounding period.

Analysis of Loan Balances

The basics of Chapter 3 and the concepts of per-payment-period interest rate can be applied to understanding the implications of loans and leasing. Consider first an amount L financed at time 0 and paid off at a per-period interest rate of i over N periods in an equal payment series (payments at $n = 1$ through $n = N$). The relevant payment is

$$A_L = L \cdot (A/P, i, N)$$

The balance immediately after the n th payment is the (time n) present value of the remaining $N - n$ payments of A_L , namely

$$P_n = A_L (P/A, i, N - n) = L \cdot (A/P, i, N) (P/A, i, N - n)$$

The interest paid at time $n + 1$ is then

$$iP_n$$

leaving a loan balance (immediately after the payment) of

$$P_n - (A_L - iP_n)$$

Analysis of Leases

In a standard auto lease, one pays a loan for a car down to an agreed upon residual value at time N , the end of the lease (at which time the lessee may either simply turn in the car or buy it for that residual price). The details of the contract are usually that

1. a down payment is required at time $n = 0$,
2. regular payments A are made at times $n = 0, 1, 2, \dots, N - 1$ (typically at the beginning of each month), and
3. often (but not always) an additional administrative fee is required at time N if the lessee opts to buy the vehicle at the end of the lease.

Analysis of Leases

If the agreed upon residual value of the leased item is R at time N , the difference between the initial price of the vehicle and the down payment (the size of the loan) is L , a per-period interest rate of i is applied to determine payments and no final administrative fee is considered, the appropriate payment can be found as follows.

For B_{N-1} the loan balance at $N - 1$ the object is to set

$$R = (1 + i) B_{N-1}$$

i.e. to have

$$B_{N-1} = (1 + i)^{-1} R$$

Analysis of Leases

Considering the initial time $n = 0$ payment the time $N - 1$ value of the amount financed is

$$(L - A)(1 + i)^{N-1}$$

and the time $N - 1$ equivalent economic value of the ($n = 1$ through $n = N - 1$) equal payment series is

$$-A(F/A, i, N - 1)$$

So to obtain the relevant payment one solves

$$B_{N-1} = (L - A)(1 + i)^{N-1} - A(F/A, i, N - 1)$$

for A . A small amount of algebra shows this to be

$$A = \frac{B_{N-1} - L}{(1 + i)^{N-1} + (F/A, i, N - 1)} = \frac{(1 + i)^{-1} R - L}{(1 + i)^{N-1} + \left[\frac{(1+i)^{N-1} - 1}{i} \right]}$$

(as written above, A is negative indicating an outflow)

Comparison of Financing Alternatives

In comparing the attractiveness of various financing options, one can simply use the basic equivalent economic values of series of cash flows considered to this point. But a very key point in doing so is that the interest rate(s) applied needs to be *not* ones somehow used under various assumptions to invent a series of cash flows (like the appropriate equal payments on a lease just discussed) but rather, **interest rate(s) reflecting the personal financial situation of the person doing the analysis!** A person with available options that would allow him or her to earn a high interest rate on money elsewhere should gladly accept a loan arrangement set up at a low interest rate in preference to an outright purchase using his or her own capital. And the tools considered thus far will show this to be the case if alternate proposed series of cash flows are subjected to the personally available interest rate(s).

Analysis of the Value of Engineering Projects

Corporate projects have costs and benefits that are realized over time and can be treated like cash flows in the evaluation of their attractiveness. For a given project one might let

$$\begin{aligned} A_n &= \text{net value produced by the project in period } n \\ &= (\text{project benefit in period } n) - (\text{project cost in period } n) \end{aligned}$$

A_0 will represent a project value at time 0 (at the start before any benefits can be enjoyed) and will typically be negative and represent an initial investment in the project. All economic analysis of engineering projects is built on the series

$$A_0, A_1, A_2, \dots$$

Conventional Payback Period

The sums

$$S_0 = A_0$$

$$S_1 = A_1 + A_0 = A_1 + S_0$$

$$S_2 = A_2 + S_1$$

and in general

$$S_n = \sum_{k=0}^n A_k = A_n + S_{n-1}$$

are the cumulative sums of net values through times n . One measure of effectiveness of a project is how quickly it "pays back" an initial investment. That is, often there is a time before which the cumulative net values are negative and after which they are positive (one might interpolate between n and $n + 1$ with $S_n < 0$ and $S_{n+1} > 0$ to produce a number stating the time location of this sign change). The time of the sign change is called the **conventional payback period**. This is very simple to compute and understand but ignores the time value of money.

Discounted Payback Period

It makes sense that in finding a payback period, the effect of interest should be considered for some appropriate rate i . (Resources invested in the project might have been borrowed at interest or could have been otherwise invested to produce interest.)

With $A_0 < 0$, at time 1 interest $-iA_0$ is owed by the project on the initial investment, so that a discounted cumulative value (a project balance) at time 1 is

$$PB_1 = A_0 + (A_1 + iA_0) = A_1 + (1 + i) A_0$$

At time 2 the corresponding discounted cumulative value (project balance) is

$$PB_2 = PB_1 + (A_2 + iPB_1) = A_2 + (1 + i) PB_1$$

and in general

$$PB_n = A_n + (1 + i) PB_{n-1}$$

Discounted Payback Period

There is often a time before which the discounted cumulative net values/project balances are negative and after which they are positive (one might interpolate between two consecutive integer times with $PB_n < 0$ and $PB_{n+1} > 0$ to produce a number between n and $n + 1$ as the time location of this sign change). The location of that sign change is called the **discounted payback period**.

While both conventional and discounted payback periods are useful descriptions of the series of values A_n , they don't discriminate between the two series

$$-5, 1, 1, 1, 1, 1, 1, 1, 1, 1$$

and

$$-500, 100, 100, 100, 100, 100, 100, 100, 100, 100$$

Other measures are needed to capture notions of project value.

Net Present Worth

So we consider equivalent economic values for a project. For i an appropriate interest rate (a required rate of return or **minimum attractive rate of return** (MARR)) and N a project service life, we consider first the time 0 equivalent economic value of the "cash flows"/net values

$$PW(i) = A_0 + A_1(1+i)^{-1} + A_2(1+i)^{-2} + \dots + A_N(1+i)^{-N}$$

the project's so-called **net present worth** (NPW for interest rate i).

$PW(0) = S_N = \sum_{n=0}^N A_n$ is the sum of the project net values across the service life and $PW(i)$ has the value A_0 as an asymptote as $i \rightarrow \infty$. Typically $PW(0) > 0$ and $A_0 < 0$ and there is a single value of i where $PW(i)$ switches sign from positive to negative. This interest rate value is called the break-even interest rate or project **rate of return**. Comparison of this rate of return to an organization's MARR indicates whether a project is expected to add value.

Net Present Worth-Two Questions and "Answers"

Exactly *what is* the net present worth of a project? If one supposes that money can be borrowed or invested at interest rate i (whether from an external source or from a company's own investment pool) and a project has (net) inflows and outflows of A_n at times n , borrowing if its balance is negative and investing if it is positive after paying the period's interest (if any), then whatever is the final project balance has a (time 0) present value. This present value is the net present worth of the project.

What goes into the setting of a MARR? The text promises to discuss this in depth in Chapter 15, but for the time being, this involves at least consideration of the cost of capital (the rate needed to make an "ordinary" project worthwhile) and any appropriate risk premium (any addition to the rate associated with project circumstances that involve unusually high risk).

Net Present Worth-Essentially Infinite Useful Life

There are projects for which the useful life is very long and (at least after some start-up variation in period net values) some constant net value or repeating pattern of net values becomes operative. It is possible to find a present net worth for such a project using a limiting value for the value of a uniform cash flow series. That is, for a uniform series

$$PW(i) = (P/A, i, N) A = \left[\frac{(1+i)^N - 1}{i(1+i)^N} \right] A$$

converges to the limit

$$CE(i) \equiv \frac{A}{i}$$

called the **capitalized cost** or **capitalized equivalent**. One way to think about this value is as the amount that would need to be invested at time 0 at rate i in order to allow perpetual (interest only) withdrawals/payments of A at each subsequent time point (always therefore having a remaining balance after withdrawal of $CE(i)$).

Other Net Worth Measures-Net Future Worth

There are other versions of the net worth idea. One can state a net future worth at time N under interest rate i as

$$FW(i) = PW(i)(1+i)^N$$

For that matter, a net future worth *any* time point n is

$$PW(i)(1+i)^n$$

Analysis Periods in Choosing Between Projects

Time, energy, and money are finite and no organization can do everything that might in the abstract be attractive. There is discussion in Chapter 5 of how one might thus identify one of several projects as most attractive on the basis of net worth considerations. The main issue that arises is finding ways to evaluate the economic effects of all projects over the same **analysis period/study period/planning horizon**. In the simplest of all cases, all projects under consideration are subject to the same required **service period** (the period established by some corporate need over which a project needs to operate) that also coincides with a common **useful lifetime** for all projects. Then the analysis period can be chosen as that common period/lifetime and comparisons can be made in terms of net present worth.

But what to do in more complicated circumstances? The text considers cases where an analysis period differs from some useful lifetime.

Where Project Lives Exceed the Length of the Study Period

Where a project life exceeds the length of a study period, one way of doing a net worth analysis is to replace all net values beyond the end of the study period with a single appropriate estimated salvage value at the end of the study period. (If, for example, a piece of equipment could be used longer than the study period, one would use a projected selling price for the item at the end of the study period.) Then an ordinary net present worth evaluation can be made on the modified series of net values. (One is effectively thinking of ending the project and liquidating its assets at the end of the study period.)

Service Projects Shorter Than the Length of the Study/Service Period

Service project alternatives are ones where the series of revenues (benefits) is the same for all (and only cost streams differ). Essentially the same work is to be produced by all possibilities over the service period, and one need not even consider the revenues in doing an economic analysis of such alternatives.

If a particular project has a useful life shorter than the desired service period, one or more replacement projects must be instituted for the remainder of the service period and their costs included in the cost analysis of the original project. (If, for example, a purchased piece of equipment has a useful life shorter than the service period, the costs associated with leasing a replacement piece for the remainder of the service period might be included in the net present worth analysis.)

Revenue Projects of Unequal Length

Revenue project alternatives are ones where both benefit/revenue and cost streams vary project to project. There is no intent for the projects to deliver the same work or streams of benefits. In this case it is often sensible to use the largest project length as the planning period. Shorter projects simply get 0 net values added to their series of cash flows after the end of their useful lifetimes. (Note that the value that they generate by the end of their lives is effectively compounded to the end of the study period in making comparisons between alternatives.)

Repeating Projects of Unequal Length

Where projects will be repeated indefinitely (e.g. vehicles bought and used up and replaced, etc.) if alternative projects have unequal lengths, one can find the least common multiple (say LCM) of the lengths and compare using repetitions of the candidate projects through time $N = LCM$. For example, if one kind of vehicle is to be replaced every 2 years and an alternative is to be replaced every 3 years, $LCM = 6$ provides for 3 cycles of the first project and 2 of the second and a sensible basis for net present worth comparisons of the two projects.

Annual Equivalent Worth

Another way of expressing the value of a project (beyond NPW and FW) is through its **annual equivalent worth**. This is the net value A that if realized yearly across the useful life of a project would produce an equivalent economic value. In the event that the analysis time scale is years to begin with, this is

$$AE(i) = PW(i)(A/P, i, N)$$

Where only costs (and not benefits) need to be considered (as in service projects) this is known as the **annual equivalent cost** $AEC(i)$.

Putting worth or cost on a "per year" basis enables a number of useful kinds of comparisons. Chapter 6 of the text is mostly a series of illustrations of this fact. In the process of making these illustrations the author introduces several financial concepts one of which we briefly record next.

Two Kinds of Costs

When doing financial analysis of a project, one must typically account for two kinds of costs: **operating costs** and **capital costs**. The former are recurring costs like those of labor and raw materials and power needed to carry on a project. The latter are incurred when equipment or other such resources are purchased for use in a project. Capital costs are usually associated with an initial (time 0) one-time **purchase cost** I and a final (time N) one-time **salvage value** S . In this framework, the annual equivalent of this pair is the **capital recovery cost**

$$\begin{aligned} CR(i) &= I(A/P, i, N) - S(A/F, i, N) \\ &= (I - S)(A/P, i, N) + iS \end{aligned}$$

here stated as a positive value (though it is a "cash" outflow to the organization). This is what an external lender will charge per year to finance the use of the equipment and what a company will pay annually for that use.

Rate of Return-Motivation

Already on slide 34 we considered the issue of establishing a "rate of return" for a project. Chapter 7 of the text returns to this topic in additional detail. The topic is motivated by analogy to very simply described investments like the purchase and subsequent sale of a security (a stock, a certificate of deposit, etc.). In these situations a purchase or deposit A_0 (a negative cash flow) at time 0 followed by a sale or withdrawal A_N (a positive cash flow) at time N has an implicit (or explicit) associated per-period interest rate i^* for which

$$A_N = -A_0 (1 + i^*)^N$$

that could be called a (per period) **rate of return** of the investment. Notice that for this situation this equation can be rewritten as

$$A_0 (1 + i^*)^N + A_N = 0$$

i.e.

$$FW(i^*) = 0 \quad \text{and/or} \quad PW(i^*) = 0 \quad \text{and/or} \quad AE(i^*) = 0$$

Rate of Return-Motivation

In the motivating case where $A_N > 0$ and $A_N > |A_0|$,
 $FW(0) = A_0 + A_1 > 0$, and in fact

$$FW(i) > 0 \text{ if } i < i^* \text{ and } FW(i) < 0 \text{ if } i > i^*$$

This makes the investment attractive exactly when a personal/available interest rate, i , is less than the "rate of return" of the investment, i^* .

Rate of return analysis is an attempt to carry this kind of thinking over to analysis of arbitrary net value series for projects. Unfortunately, the simplicity of the basic two-flow investment situation doesn't necessarily carry over to the analysis of more complex series of cash flows/net values $A_0, A_1, A_2, \dots, A_N$. Perhaps the best way to think about the content of the text's Chapter 7 is as an exposition of what can be salvaged in terms of a rate of return concept for evaluating the attractiveness of a single project and comparing different potential projects.

Rate of Return-Technical Definition

The net present worth of a project, $PW(i)$, as defined on slide 34 is a polynomial in the variable $(1+i)^{-1}$ with coefficients $A_0, A_1, A_2, \dots, A_N$. (Equivalently the net future, $FW(i) = PW(i)(1+i)^N$, is a polynomial in the variable $(1+i)$ with coefficients $A_N, A_{N-1}, A_{N-2}, \dots, A_0$.)

Essentially, rate of return analysis amounts to consideration of roots of these polynomials.

That is, reasoning by analogy to the motivating cases, a rate of return (ROR) or an "internal" rate of return (IRR) of a project is an interest rate i^* for which

$$PW(i^*) = 0$$

Equivalently, an IRR makes the present value of positive cash flows/net values equal in magnitude the present values of negative cash flows/net values. It's an interest rate that makes a project's terminal (time N) balance 0.

When is Interpretation of IRR Clear?

It's just a mathematical fact that for arbitrary sequences of net values the equation, $PW(i) = 0$ can have multiple positive roots. In those cases as one increases i from 0, $PW(i)$ switches from positive to negative to positive etc. *and there is no hope* of correctly saying anything like "when external rates available are less than an i^* , the project is attractive." So (in order to salvage the basic IRR motivation) a fair amount of energy is expended trying to identify cases where there is only one IRR (and its interpretation is thus clear).

Methods of studying the behavior of the function $PW(i)$ (and location of any roots of $PW(i) = 0$) include

- plotting of $PW(i)$ versus i and numerical solution of $PW(i) = 0$,
- analytic (pencil and paper) solution of $PW(i) = 0$ (this is rarely feasible, as polynomial equations beyond quadratics are not so easy to solve), and
- application of theorems about roots of polynomials.

Rate of Return and Simple and Non-Simple Projects

Descartes' "rule of signs" applied to a net value series guarantees that there will be at most as many IRR's as there are sign changes in the series (ignoring 0's in the series). This motivates the language that a project is **simple** if it has exactly one change of sign in the series $A_0, A_1, A_2, \dots, A_N$. A simple project with $A_0 < 0$ can be called a **simple investment**. A simple project with $A_0 > 0$ can be called a **simple borrowing**. Simple projects (by the "net cash flow rule of signs") have no more than one IRR and such an IRR thus has a clear interpretation.

Only non-simple projects can have multiple IRR's. However, some non-simple projects have only a single IRR. One easy way of establishing that even a non-simple project has at most one IRR is to use the "accumulated cash flow sign test" (based on "Norstrom's criterion"). This says that if a cumulative net value series S_0, S_1, \dots, S_N (see slide 31 for the notation S_n) has $S_0 < 0$ and exactly one sign change, there is one IRR (that thus has a simple interpretation).

Project Balances and Net or Pure Investments

The project balance idea first used on slide 32 is relevant in discussing IRRs. With $PB_0 = A_0$ and

$$PB_n = PB_{n-1} (1 + i) + A_n$$

if $A_0 < 0$ and for each IRR each $PB_n \leq 0$ for all $n < N$, the text calls the project a **pure** investment or **net** investment. (A project with $A_0 > 0$ and for each IRR i^* , each $PB_n \geq 0$ for all $n < N$ is called a **pure** borrowing.) For pure projects and IRR's i^* , project balances are first 0 at time N . It turns out that a *simple* investment (borrowing) is also a *pure* investment (borrowing).

Projects with $A_0 < 0$ that are not pure are called **mixed** investments. For some $n < N$ and i^* they have $PB_n > 0$ and the company acts as a borrower from the project rather than as an investor during the next period.

Project Balances and Internal and External Interest Rates

Another notion of ROR involves using different interest rates for positive and negative project balances, PB_n . If i is a rate internal to a company representing the value that the company produces using its resources and $MARR$ is a rate available externally (and typically smaller than i), then project balances might be computed using $PB_0^\# = A_0$ and

$$PB_n^\# = \begin{cases} PB_{n-1}^\# (1 + i) + A_n & \text{if } PB_{n-1}^\# < 0 \\ PB_{n-1}^\# (1 + MARR) + A_n & \text{if } PB_{n-1}^\# > 0 \end{cases}$$

Then an interest rate i making

$$PB_N^\# = 0$$

might be called a **true IRR** or **return on invested capital (RIC)**. For a pure investment, at an IRR all $PB_n^\# = PB_n$, so that IRR is also an RIC. This development can be understood as pointing out the "internal" nature of an IRR for a simple investment that is therefore pure and has a single IRR.

Rate of Return and Comparing Alternative Projects

Even in the elementary motivating context of the purchase and subsequent sale of securities, rates of return don't provide good direct guidance in choosing between alternative projects. At the end of the day, the text recommends instead use of total/absolute dollar measures like NPW, NFW, and AEW for choosing among projects. But if some analysis based on ROR is insisted upon, it is essential to make use of so-called **incremental** methods that consider not raw net value series but rather differences of such series.

Suppose two projects have respective net value series $A_0, A_1, A_2, \dots, A_N$ and $B_0, B_1, B_2, \dots, B_N$ where we will suppose that $B_0 < A_0 < 0$ (the "A" project has the lower initial investment cost). To begin, one requires that both have IRR's at least as big as the MAAR. (Otherwise the offending project is not considered.)

Rate of Return and Comparing Alternative Projects

For purposes of comparing the two projects, one then computes the series of differences

$$B_0 - A_0, B_1 - A_1, B_2 - A_2, \dots, B_N - A_N$$

and attempts to find a meaningful IRR for the difference series, say IRR_{B-A} . Then the "B" project is preferable to the "A" project exactly when

$$IRR_{B-A} > MARR$$

Where more than two projects are under consideration, one treats them in pairs (ordering by size of initial investment, in each comparison eliminating one) followed by comparing the two projects remaining with the largest initial investments until only one remains.