Engineering Economy Outline

IE 305-Part 2

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Kinds of Production Costs

Costs incurred in production can be classified in many helpful ways.

- Manufacturing costs include *direct* raw material and labor costs and manufacturing *overhead* (like machine maintenance costs).
    - Manufacturing costs of materials include those of raw materials awaiting use, those in process, and those in finished inventory awaiting sale and delivery.
- Non-manufacturing costs include general company overhead, sales costs, and administrative costs.

Costs incurred generating particular revenue are recognized as expenses in the period that the revenue is recognized.

- Period costs are matched on a time period basis (and are usually non-manufacturing costs).
- Product costs are matched on the basis of output produced (and are often manufacturing costs).

Costs and Production Volume

Amount of production accomplished is quantified in terms of a **volume index**, like number of parts or assembled units or gallons of output. How different costs vary according to the value of such an index is important. **Fixed costs** are those associated with maintaining a basic operating capacity and don't change with the volume index. **Variable costs** depend upon the level of the volume index. Some costs are **mixed costs** in that they are constant for small values of the volume index and then increase after some threshold level of the index is crossed.

The **average unit cost** for some production is just that, namely

$$\frac{\text{cost of production}}{\text{corresponding value of the volume index}}$$

Formal and Informal Cost Considerations for Decision-Making

When making business decisions (choosing between opportunities) one always considers *differences* in total costs between alternatives, i.e. **differential costs**. These are considered only *going forward/in the future*, i.e. ignoring **sunk costs** that are in the past and cannot be changed.

As one considers business (or life) opportunities, even if no formal analysis is done, account should be taken of **opportunity costs**. These are potential *benefits implicitly given up* when one does something, because finite energy, time, resources then dictate that one must then fail to do other things.

A **marginal cost** associated with production is the cost of producing one additional unit of the volume index (based on a particular level of the value index). (This is effectively the derivative of the production-cost-versus-volume-index curve.)

Linear Production Costs

At least as an approximation to reality, it is often useful to model production costs as linear in the volume index. That is, for

$$F = \text{a fixed production cost}$$

and (a constant in the volume index)

$$v = \text{variable production cost per unit}$$

for volume index, x, the production cost is

$$F + vx$$

and

$$\text{average unit cost} = \frac{F + vx}{x} = v + \frac{F}{x}$$

Profit for Linear Production Costs and Constant Selling Price

Then if

$$p = \text{selling price per unit}$$

is fixed, i.e. doesn't change with

$$x = \text{sales volume}$$

for Φ a **profit from production**

$$\begin{aligned}\Phi &= \text{sales revenue} - \text{production cost} \\ &= px - (F + vx) \\ &= -F + (p - v)\,x\end{aligned}$$

Linear Production Costs and Constant Selling Price (continued)

The slope of the linear relationship between sales volume and profit is

$$p - v = \text{the (per unit) contribution to margin}$$

(or **marginal contribution**).

The **marginal contribution rate** is

$$\frac{p-v}{p} = 1 - \frac{v}{p}$$

The **break-even volume** is the value of x that produces $\Phi = 0$, namely

$$x_{\text{b}} = \frac{F}{p-v}$$

that can be expressed in dollars as

$$px_{\text{b}} = \frac{pF}{p-v} = \frac{F}{1-\frac{v}{p}}$$

(the fixed cost over the marginal contribution rate).

Linear Production Costs and Constant Selling Price (continued)

Various problems can be set based on the basic linear relationship between sales volume and profit

$$\Phi = -F + (p - v)\, x$$

The equation has five entries and one can give any four and solve for the fifth. And one can contemplate the effects on profit of changing any of the values F, p, v, and x. In this regard, if one likes large profit, one likes small F (one near 0 rather than being a large positive number), large x, large p, and small v. Of course, in practice, not all of these can be varied independently (e.g. increasing price substantially would be expected to reduce sales volume).

Depreciation Generalities

Corporate assets that wear out over time are usually subject to "depreciation." This is a calculated reduction in the value of the asset. The two main purposes of this calculation are:

1. accurate accounting for the value of assets held by a company (for both business planning and fair representation of the state of the company to stockholders and others) and
2. computation of corporate income taxes–that can be reduced by using depreciation amounts as business expenses that offset business income.

In order to be subject to depreciation, assets must

1. be used in business or be held for sale,
2. have a definite useful life longer than 1 year, and
3. wear out, lose value, or become obsolete with time.

Depreciation Generalities

In order to calculate depreciation values, one must specify

1. a **cost basis**,
2. a **depreciable life**,
3. an **estimated/predicted salvage value**, and
4. a **depreciation method**.

The cost basis of an asset includes its purchase price, plus delivery and installation (and other such) charges. If the asset is a replacement for another, this figure can be reduced by any "unrecognized gain" that is the difference between the salvage value of the old asset and its **book value** (its current depreciated value).

A salvage value for an asset is estimated when it is acquired and this estimate is the basis for some kinds of depreciation calculations. Its realized/actual value is later used in tax calculations at asset disposal.

Depreciation Generalities

Depreciation methods consist of

1. those used for setting book values (the depreciated values) of the assets that a business owns, and
2. those used in determining corporate income taxes.

Depreciable lives are set by a firm (possibly using published government guidelines) for book depreciation and are specified by law for tax depreciation. In the case of current US tax depreciation, all assets that can be depreciated for tax purposes fall into classes with 3-,5-,7-,10-,15-, 20-,27.5-, or 39-year depreciation lives.

Book Depreciation Methods-Units of Production and Straight Line methods

In some cases an asset has a useful lifetime that can be measured in terms of units of production (or volume index). Here for a cost basis I, and estimated salvage value S_{est}, depreciation can be computed as

$$\begin{aligned} D_n &= \text{depreciation for year } n \\ &= \frac{\text{number of service units used in year } n}{\text{total number of service units in the life of the asset}} \left(I - S_{\text{est}}\right) \end{aligned}$$

For I and S_{est} as above and a depreciable/useful life of N years, the **straight line** depreciation method sets each D_n at

$$D_n = D \equiv \frac{I - S_{\text{est}}}{N}$$

Sum of Year Digits Method

Methods that take larger depreciation amounts early in the life of an asset and smaller ones later are called **accelerated methods**. One old such method is the **sum of year digits** method. This sets the depreciations proportional to

$$N, N-1, N-2, \ldots, 1$$

in years

$$1, 2, 3, \ldots, N$$

Since

$$1+2+3+\cdots+N = \frac{N(N+1)}{2}$$

for I, S_{est}, and N as before, the sum of year digits depreciation is thus

$$D_n = \frac{N+1-n}{1+2+3+\cdots+N}\left(I - S_{\text{est}}\right) = \frac{2(N+1-n)}{N(N+1)}\left(I - S_{\text{est}}\right)$$

Declining Balance Method

The most common accelerated depreciation method is a kind of geometrically decreasing depreciation method called the **declining balance** method. The simplest version of this ignores any predicted salvage value S_{est} and uses a multiplier

$$0 < \alpha \leq \frac{2}{N}$$

For B_n the book value (balance) at the end of year n, the simple declining balance depreciation is

$$D_n = \alpha B_{n-1}$$

Simple algebra shows that under this scheme

$$B_n = I(1-\alpha)^n \quad \text{and thus} \quad D_n = \alpha (1-\alpha)^{n-1} I$$

The $\alpha = 2/N$ case is called the "200%" or "double" declining balance case and $\alpha = 1.5/N$ is the "150%" declining balance case.

Declining Balance Method-Adjustments for Salvage Value

Note that the simple final declining balance book value is $B_N = I(1-\alpha)^N$ and unless a predicted salvage value is (fortuitously) this, one ends the life of the asset with a (positive) book value that differs from the predicted salvage value. So adjustments are typically made to the method to account for this discrepancy.

Where $I(1-\alpha)^N$ exceeds a predicted salvage value, a common modification of ordinary declining balance depreciation is to switch to straight line depreciation in the first year n where straight line depreciation of the current book value over the remaining life of the asset is advantageous. That is, one switches at the first n for which

$$\frac{B_{n-1} - S_{\text{est}}}{N+1-n} = \frac{I(1-\alpha)^{n-1} - S_{\text{est}}}{N+1-n} > \alpha B_{n-1}$$

Call this first year under straight line depreciation year n_{switch}.

Declining Balance Method-Adjustments for Salvage Value

That is, where $I(1-\alpha)^N > S_{\text{est}}$ it is common to take depreciations

$$D = \frac{B_{(n_{\text{switch}}-1)} - S_{\text{est}}}{N+1-n_{\text{switch}}} = \frac{I(1-\alpha)^{(n_{\text{switch}}-1)} - S_{\text{est}}}{N+1-n_{\text{switch}}}$$

in each of years $n_{\text{switch}}, n_{\text{switch}}+1, \ldots, N$.

Where $I(1-\alpha)^N < S_{\text{est}}$, one must simply stop short of depreciating the asset beyond its predicted salvage value. This means taking only the depreciation

$$B_{n-1} - S_{\text{est}}$$

in the year n that one first would have $B_n = I(1-\alpha)^n < S_{\text{est}}$. Call this year n_{last}. Depreciations are then $D_{n_{\text{last}}} = B_{n_{\text{last}}-1} - S_{\text{est}}$ and 0 in years $n_{\text{last}}+1, \ldots, N$.

(US Federal) Tax Depreciation Methods

Depreciation is an important topic for business viability since depreciation of corporate assets can be used to offset revenue, reduce taxable income, and thereby contribute to corporate *net* income. It is thus important to know something about the basic rules for tax depreciation in the US.

Assets put into service by US companies before 1981 are depreciated using book methods (SL, SOYD, or DB). Assets put into service 1981-1986 must be depreciated using an "Accelerated Cost Recovery System" specified in tax law. Assets placed in service after 1986 are treated using a **Modified Accelerated Cost Recovery System** (**MACRS**) specified in tax law. The basics of this last system/set of rules bear presentation in the next few slides. They are based on application of book method principles in a way intended to make corporate investment attractive by allowing much of an asset's depreciation to be taken early in its life.

MACRS-Property Classes

All corporate assets depreciable under the MACRS are placed (by regulation) into 3-,5-,7-,10-,15-, or 20-year classes for "personal" properties (assets that are not real estate) and 27.5- or 39-year classes for "real" (real estate/land and buildings) properties.

MACRS depreciation on 3-,5-,7-, 10-, 15- and 20-year (personal) properties is computed using

1. a special "half year" convention that treats properties as if they were acquired half way through a first year of ownership and used up 3,5,7,10 or 15 years later (in year 4,6,8,11, or 16 of ownership) and thus spreads depreciation over one more year than the ownership class number,
2. the 200% declining balance method switching to the straight line method (when it first provides a larger depreciation) for 3-,5-,7-, and 10-year properties, and
3. the 150% declining balance method switching to the straight line method for 15- and 20-year properties.

MACRS Table for "Personal" Properties

Here is the IRS table specifying percent of cost basis that can be taken as depreciation for 3- through 20-year personal properties.

Table A-1. **3-, 5-, 7-, 10-, 15-, and 20-Year Property Half-Year Convention**

Year	Depreciation rate for recovery period					
	3-year	5-year	7-year	10-year	15-year	20-year
1	33.33%	20.00%	14.29%	10.00%	5.00%	3.750%
2	44.45	32.00	24.49	18.00	9.50	7.219
3	14.81	19.20	17.49	14.40	8.55	6.677
4	7.41	11.52	12.49	11.52	7.70	6.177
5		11.52	8.93	9.22	6.93	5.713
6		5.76	8.92	7.37	6.23	5.285
7			8.93	6.55	5.90	4.888
8			4.46	6.55	5.90	4.522
9				6.56	5.91	4.462
10				6.55	5.90	4.461
11				3.28	5.91	4.462
12					5.90	4.461
13					5.91	4.462
14					5.90	4.461
15					5.91	4.462
16					2.95	4.461
17						4.462
18						4.461
19						4.462
20						4.461
21						2.231

Figure: From IRS Publication 946

MACRS for Real Properties and Depletion

The MACRS rules for depreciating the buildings part (NOT THE LAND PART) of a depreciable real estate asset include

1. a special "half month" convention that treats properties as if they were acquired half way through a first month of ownership and used up 27.5 years (for residential properties) or 39 years (for commercial properties) later and thus spreads depreciation over one more year than the ownership class number,
2. straight line depreciation over $27.5 \times 12 = 300$ months for residential real properties, and
3. straight line depreciation over $39 \times 12 = 468$ months for commercial real properties.

Another topic discussed in the text's Ch9 is **depletion**. This is the physical reduction of an initial supply of natural resources (like minerals or timber). The tax rules for subtracting the effects of depletion from revenue are different from depreciation rules.

Corporate Income Tax

Companies in the US are taxed on

$$taxable\ income = gross\ income - expenses$$

(where expenses include cost of goods sold, depreciation, and operating expenses) producing

$$net\ income = taxable\ income - taxes$$

Taxable income is taxed (as for persons) more or less "progressively," with the "last" dollars earned in a year usually taxed at a higher rate than the "first" ones. **Marginal tax rates** refer to rates applied to the last dollars. An **effective (or average) tax rate** is

$$\frac{total\ tax}{taxable\ income}$$

US Corporate Tax Rates

Here is the IRS table for 2012 corporate taxes

Tax Rate Schedule

If taxable income (line 30, Form 1120) is:			
Over—	**But not over—**	**Tax is:**	**Of the amount over—**
$0	50,000	15%	-0-
50,000	75,000	$ 7,500 + 25%	$50,000
75,000	100,000	13,750 + 34%	75,000
100,000	335,000	22,250 + 39%	100,000
335,000	10,000,000	113,900 + 34%	335,000
10,000,000	15,000,000	3,400,000 + 35%	10,000,000
15,000,000	18,333,333	5,150,000 + 38%	15,000,000
18,333,333	—	35%	-0-

Figure: From IRS Publication 542

Disposal of an MACRS Asset-Final Book Value, Gain or Loss

The first matter to be considered in the tax implications of disposal of an MACRS asset is the asset's final book value. If disposal occurs *after* the end of the depreciation period, this is 0. If disposal occurs *before* the end of the depreciation period, only a half year depreciation is subtracted from the previous book value to arrive at the book value (in the final year of depreciation, this brings the book value to 0).

After finding a book value for the disposal year, there is the matter of the tax implications of a gain or loss on the property. By virtue of depreciation

$$\textit{final book value} < \textit{cost basis}$$

and we consider in turn the possibilities of where the actual salvage value (proceeds from sale minus selling and removal expenses, call it S_{act}) lies in comparison to the last book value and original cost basis. There is typically an associated gain or loss.

Disposal of an MACRS Asset-S Less Than Book Value

Where

$$S_{\text{act}} < \textit{final book value}$$

the difference

$$S_{\text{act}} - \textit{final book value}$$

is negative, representing a **loss**. This can be used to offset revenue in the tax year. If the marginal tax rate for the company is t_m (and this calculation doesn't move income past a change point in the tax schedule) this produces a (positive) **tax saving** of

$$t_{\text{m}} \left| S_{\text{act}} - \textit{final book value} \right|$$

and makes the **net proceeds from the sale** of the asset

$$S_{\text{act}} + t_{\text{m}} \left| S_{\text{act}} - \textit{final book value} \right|$$

Disposal of an MACRS Asset-S More Than Book Value

Where

$$S_{\text{act}} > \textit{final book value}$$

the difference

$$S_{\text{act}} - \textit{final book value}$$

is positive, representing a **gain**.

If additionally $S_{\text{act}} < \textit{cost basis}$ then *all* of the gain is an **ordinary gain**.

If additionally $S_{\text{act}} > \textit{cost basis}$ then

$$S_{\text{act}} - \textit{cost basis} = \textbf{capital gain}$$

and

$$\textit{cost basis} - \textit{final book value} = \textbf{ordinary gain}$$

Ordinary gains are also known as **depreciation recapture**.

Disposal of an MACRS Asset-Gains Tax

Where there are gains on the disposal of MACRS assets, there is tax to pay on them. Presently, ordinary and capital gains are taxed at the same rate, as ordinary corporate income. But they must be kept separate for accounting purposes, one reason being the fact that tax rules can be changed at any time in the future, and capital and ordinary gains are typically considered to be fundamentally different.

If the marginal tax rate for the company is t_{m} (and this calculation doesn't move income past a change point in the tax schedule) this produces a **gains tax** of

$$\begin{aligned} gains\ tax &= ordinary\ gains\ tax + capital\ gains\ tax \\ &= t_{\text{m}} \left(ordinary\ gain \right) + t_m \left(capital\ gain \right) \end{aligned}$$

(where there may be 0 capital gain). The corresponding **net proceeds** from the sale of the asset are

$$S_{\text{act}} - gains\ tax$$

Combined Federal and State Tax Rate

If

$$t_f = \textit{federal marginal tax rate}$$
$$t_s = \textit{state marginal tax rate}$$

and one computes and subtracts the (lower) state tax from income before computing federal tax, it's easy to see that with

$$t_m = \textit{combined marginal tax rate}$$

one has

$$t_m = t_f + t_s - t_f t_s$$

Depreciation, Taxes, Net Income, and Cash Flows

Depreciation is *not* a cash flow. No money changes hands, nor are resources invested when depreciation is accounted for. But it is included on income statements as an expense offsetting revenue and reducing tax liabilities. It thus indirectly affects project cash flows in that it reduces taxes paid, real cash outflows.

Where one wishes to determine a project cash flow from a corresponding net income, any depreciation expense that has been subtracted from revenue in computing the net income should be added back in to produce the proper cash flow figure.

Cash flows are a more useful representation of project effectiveness than project net incomes, since the latter ignore the actual timing of cash inflows and outflows (counting costs used to produce revenues only when the revenues are realized) and thus also ignore the time value of money.

Tax Rate to Use for Project Analysis

When one wants to find a tax rate relevant to a potential engineering project, the appropriate rate is an average rate based on incremental taxable income and incremental tax. That is, the appropriate rate is

$$\frac{\textit{tax with the project} - \textit{tax without the project}}{\textit{taxable income with the project} - \textit{taxable income without the project}}$$

Types of Project Cash Flows

Project cash flows are usually thought of as derived from the broad categories of

1. **Operating Activities** (including revenues, expenses, project debt interest, and income taxes),
2. **Investing Activities** (including project investments, net proceeds from salvage of project assets, and investments in working capital), and
3. **Financing Activities** (including proceeds from loans taken for the project and repayments of principal on such loans).

Notation

In what follows, let

$$\begin{aligned} A_n &= \text{project cash flow at time } n \\ R_n &= \text{project revenue at time } n \\ E_n &= \text{project expenses}^1 \text{ at time } n \\ IP_n &= \text{project debt interest at time } n \\ T_n &= \text{project income taxes at } n \\ I_n &= \text{project investment at time } n \\ S_n &= \text{project total (actual) salvage proceeds at time } n \\ G_n &= \text{project gains tax at time } n \\ W_n &= \text{project working capital investment at time } n \\ B_n &= \text{proceeds from a project loan at time } n \\ PP_n &= \text{project loan principal repayments at time } n \end{aligned}$$

[1] depreciation and debt interest are not included here

Cash Flow Development

In the notation on the previous slide

$$
\begin{aligned}
A_n = R_n - E_n - IP_n - T_n & \qquad \text{(from } \textbf{Operating activity}\text{)} \\
- I_n + (S_n - G_n) - W_n & \qquad \text{(from } \textbf{Investment activity}\text{)} \\
+ B_n - PP_n & \qquad \text{(from } \textbf{Financing activity}\text{)}
\end{aligned}
$$

In the event that a project does not change a corporate marginal tax rate t_{m}, this can be rewritten as

$$
\begin{aligned}
A_n = & \left(R_n - E_n - IP_n\right)\left(1 - t_{\text{m}}\right) + t_{\text{m}} D_n \\
& - I_n + (S_n - G_n) - W_n \\
& + B_n - PP_n
\end{aligned}
$$

and the term $t_{\text{m}} D_n$ is sometimes called the **depreciation tax shield**.

Working Capital and Loans

The "working capital" (W_n) elements of the cash flow development concern funds that must be dedicated to cash, accounts receivable, and inventory to make a project work. For example, raw materials must be owned, as must work in progress and finished goods awaiting sale. This is accounted for by negative cash flows that are matched by later positive cash flows as (years end or) the project ends and the corresponding capital is returned to the company.

We further reiterate that project loans (B_n) *are not* income and repayments of principal on them (PP_n) are not expenses. They have no tax consequences. But *they are* cash flows and have economic consequences.

Inflation is a second (besides interest) aspect of the time value of money. It accounts for the fact that prices for goods and services change with time. A dollar today may not have the same purchasing power tomorrow. **Inflation** occurs when prices rise/purchasing power of currency declines. **Deflation** occurs in the (fairly rare case) when prices decline/purchasing power of currency rises.

Governments must try to measure inflation/deflation. The US government produces the Producer Price Index (**PPI**) that attempts to (monthly) measure industrial prices. It also produces several versions of the Consumer Price Index (**CPI**) that attempts to measure the current price of a "market basket" of goods and services typically purchased by urban residents/workers. Official information about the current version of the latter can be found at: http://www.bls.gov/news.release/pdf/cpi.pdf

CPI Details

An old version of the CPI used average prices in 1967 as a base, while new versions[2] of the CPI use 1982-1984 average prices as a base. These 1982-1984 average prices in various categories were set to 100 (percent) and weighted according to what fraction of an average income was spent on goods or services in a given category. In subsequent years multiples of 100 representing category average prices relative to the base years are weighted together to produce a CPI. So, in theory, the ratio

$$\frac{CPI \text{ in year } n+k}{CPI \text{ in year } n}$$

is the ratio of "the market basket" prices in the two years n and $n+k$.

[2] There are versions of the CPI that intend to describe an average US urban resident (CPI-U) and an average US urban wage earner/clerical worker (CPI–W).

CPI Reweighting

A fundamental/inescapable difficulty involved in producing any price index (certainly including the CPI) is that products and patterns of their use change over time. (Indeed some products will pass out of existence and new ones will be created.) So periodic reweighting of the components of the CPI is absolutely inevitable.

But it is a politically sensitive question *how often* reweighting should be done. This is because many US government payments (like social security benefits) are tied to the value of the CPI, and frequent updating can be argued to ultimately exert pressure on consumer behavior toward substitution of cheaper alternatives in place of more expensive ones and understating of the real effects of inflation (and reduction in cost-of-living adjustments to benefits). "**Chained**" versions of the CPIs use very frequent (monthly) reweighting.

Average Annual Inflation Rate

An inflation rate f such that for a good of interest

$$\frac{\text{year } n+k \text{ price of the good}}{\text{year } n \text{ price of the good}} = (1+f)^k$$

is an average annual inflation rate for the good over the k years $n, n+1, n+2, \ldots, n+k$.

A rate $\bar{f}$ such that

$$\frac{CPI \text{ in year } n+k}{CPI \text{ in year } n} = (1+\bar{f})^k$$

is the **general (annual) inflation rate** across the k years $n, n+1, n+2, \ldots, n+k$. Put differently, the general inflation rate from year n to year $n+k$ is

$$\bar{f} = \left(\frac{CPI \text{ in year } n+k}{CPI \text{ in year } n}\right)^{1/k} - 1$$

Inflation-Actual and Constant Dollars

In the analysis of engineering projects **actual dollars** are just that, government-printed paper certificates that are used to pay debts. When we use them in cash flow analysis, we are assuming that future inflation or deflation will do whatever they will do to purchasing power. On the other hand, **constant dollars** are estimates of future cash flows in terms of the purchasing power of actual dollars in some base year.

For $\bar{f}$ a (supposedly constant) general inflation rate, a time n cash flow in (time 0) constant dollars A'_n is related to a corresponding to a time n cash flow in actual dollars, A_n, by

$$A_n = A'_n \left(1 + \bar{f}\right)^n$$

Equivalently

$$A'_n = A_n \left(1 + \bar{f}\right)^{-n}$$

Inflation and Economic Equivalence Calculations

There are two possibilities for doing economic equivalence calculations:

1. One may state cash flows in *actual dollars* and use an interest rate, i, that is a "**market**" or "**inflation adjusted**" **interest rate** that is what one can get from a financial institution on the open market and takes into account the combined effects of the earning value of capital and changes in purchasing power over time, or
2. one may state cash flows in *constant dollars* and use an interest rate, i', that is a "**real**" or "**inflation free**" **interest rate** that describes the true earning power of money with inflation/deflation effects removed.

Since taxes are paid in actual dollars, private sector economic equivalence calculations are usually done as in 1. Since governments pay no taxes, economic equivalence analyses for long-term public projects are often done as in 2.

Constant Dollar Economic Equivalence Analysis

To conduct a constant collar analysis one may convert cash flows A_n to constant dollar cash flows A'_n via

$$A'_n = A_n \left(1 + \bar{f}\right)^{-n}$$

and then consider corresponding time $n = 0$ net present worth under a real interest rate i'

$$\begin{aligned} A'_n \left(1 + i'\right)^{-n} &= A_n \left(\left(1 + \bar{f}\right)\left(1 + i'\right)\right)^{-n} \\ &= A_n \left(1 + \bar{f} + i' + \bar{f} i'\right)^{-n} \end{aligned}$$

Note that this is completely equivalent to doing analysis with actual dollar cash flows and with market rate

$$i = \bar{f} + i' + \bar{f} i'$$

computing time $n = 0$ net present worth.

Market and Real Interest Rates

The relationship between interest rates

$$i = \bar{f} + i' + \bar{f}\,i'$$

implicit in constant dollar economic equivalence computations suggests how a real interest rate might be computed. That is, solving the above for i' one obtains

$$i' = \frac{i - \bar{f}}{1 + \bar{f}} = \frac{1 + i}{1 + \bar{f}} - 1$$

Effects of Inflation on Projects

Inflation has the effects on a project of

1. causing taxable income to be over-stated because **depreciation** is in actual dollars that are devalued in time,
2. increasing taxes because inflated actual **salvage values** are compared to uninflated book values,
3. reducing real financing costs because **loans** are repaid in (devalued) actual dollars,
4. increasing **working capital** costs (a phenomenon called working capital drain),
5. reducing **net present worth** in light of 1, 2, and 4 above unless revenues are increased to keep pace with inflation, and
6. similarly reducing **IRR** unless revenues are increased to keep pace with inflation.

Incomplete Knowledge of the Future

Measures of project economic impact like net present worth and internal rate of return are functions of all cash flows $A_0, A_1, \ldots, A_N$ and period interest rates $i_1, i_2, \ldots, i_N$. We have just discussed the truth that each A_n is itself a function of period variables R_n, E_n, IP_n, D_n, I_n, S_n, W_n, B_n, PP_n, and applicable tax rates. When analysis of the probable impact of a project is done, values of these variables are in the future, and of necessity incompletely known. One takes one's best guess at them, but surely doesn't expect to get them all exactly right. The text's Chapter 12 attempts to consider means of accounting for and measuring the risk inherent in one's incomplete knowledge of the many inputs to an NPW or IRR computation. These are **break-even analysis**, **scenario analysis**, **sensitivity analysis**, and **probabilistic analysis**.

Break-Even Analysis and Scenario Analysis

We will henceforth call a set of best guesses at the inputs to a NPW computation a **base case** of inputs. If one then picks one of the many inputs for analysis, call it x for the time being, it's possible to treat NPW as a function of that single input alone (by holding the rest of the inputs at their base values). A **break-even point** for x is a value x_{b} that (at the base values of the other inputs) makes $NPW = 0$. Typically, NPW is negative to one side of x_{b} and positive to the other. One exactly breaks even at $x = x_{\text{b}}$.

In addition to a base case of inputs one might also specify a **best case** (consisting of inputs presumably most favorable to NPW) and a **worst case** (consisting of inputs presumably least favorable to NPW). Comparing the corresponding NPWs (or IRRs) for the base, best and worse **scenarios** gives some understanding of the extremes of what value the project could ultimately produce.

Sensitivity Analysis

For every input to a NPW (or IRR) computation, x with base value x_{base}, one might compute (say)

$$.9x_{\text{base}}, .95x_{\text{base}}, x_{\text{base}}, 1.05x_{\text{base}}, 1.1x_{\text{base}}$$

and corresponding values of NPW (or IRR) for these values of x with the other inputs held at their base values. Then plotting points $\left(.9, NPW_{.90x_{\text{base}}}\right)$, $\left(.95, NPW_{.95x_{\text{base}}}\right)$, $\left(1, NPW_{x_{\text{base}}}\right)$, $\left(1.05, NPW_{1.05x_{\text{base}}}\right)$, $\left(1.1, NPW_{1.10x_{\text{base}}}\right)$ one might consider the slope at 1 of the "curve" they define, and compare slopes across the various inputs. Large (positive or negative) slope of a plot at 1 indicates that NPW (or IRR) is relatively sensitive to (percent) changes in the corresponding input variable. Nearly horizontal plots identify inputs that (at least near the base case) don't much change NPW (or IRR).

Probabilistic Analysis-Distributions for Inputs

A different approach to the analysis of uncertainty in NPW (or IRR) associated with incomplete knowledge of its inputs is through providing a probability distribution for the set of the inputs. That is, letting **inputs** stand for the whole set of quantities needed to produce NPW (or IRR), if one treats **inputs** as random, then

$$NPW\left(\mathbf{inputs}\right)$$

is a random variable. A (joint) distribution for **inputs** produces a distribution for NPW.

Convenient joint distributions for **inputs** treat individual inputs as independent with means equal to respective base values. The levels of uncertainty in the inputs can be reflected in magnitudes of standard deviations of the inputs (used in the joint distribution). But how to go from this kind (or any kind) of distribution for **inputs** to the implied distribution for $NPW\left(\mathbf{inputs}\right)$ is a separate question.

Probabilistic Analysis-Simulations

The most practical method of finding (at least approximately) the distribution of NPW (**inputs**) is through **simulation** of a large number of realizations of the whole set of inputs, say

$$\mathbf{inputs}_1, \mathbf{inputs}_2, \ldots, \mathbf{inputs}_K$$

and direct computation of the corresponding realizations of NPW (or IRR)

$$NPW\left(\mathbf{inputs}_1\right), NPW\left(\mathbf{inputs}_2\right), \ldots, NPW\left(\mathbf{inputs}_K\right)$$

Properties of this set of simulated NPW values serve as approximations for the theoretical properties of the random variable NPW (**inputs**). A relative frequency distribution of the generated values approximates the probability distribution of the random variable, the sample mean approximates $\mathrm{E}NPW$ (**inputs**), and the sample variance approximates $\mathrm{Var}NPW$ (**inputs**).

Probabilistic Analysis-Analytical Methods, Mean and Variance

The text has some very artificial/simplified/unrealistic examples of using pencil-and-paper theoretical derivations of a probability distribution for NPW (**inputs**) and the mean and variance for this variable where only one or two of its inputs are treated as random.

What is perhaps more important for IE 305 purposes is to simply recall how to get the mean and variance for a discrete random variable from its distribution and to apply those ideas to problems where a distribution for NPW is been given. If the random variable NPW has possible values $npw_1, npw_2, \ldots, npw_M$ with corresponding probabilities $p_1, p_2, \ldots, p_M$ then the mean and variance of the random variable are respectively

$$\mathrm{E}NPW = \sum_{i=1}^{M} npw_i p_i \quad \text{and} \quad \mathrm{Var}NPW = \sum_{i=1}^{M} \left(npw_i - \mathrm{E}NPW\right)^2 p_i$$

Probabilistic Analysis-Comparison of Projects

When comparing projects under uncertainty, one generally prefers projects with large ENPW. But relatively small VarNPW is also typically desirable, in that small variance can be taken as relative certainty regarding the economic consequence of a project. So small differences between *mean* NPWs (or IRRs) can be unimportant if projects have wildly different *standard deviations* of NPW (and thus varied uncertainties regarding the benefits of the projects).

If one has available the sets of possible values of NPW and corresponding probabilities for two potential projects, another way to compare them is to treat NPW_{A} and NPW_{B} as independent random variables and find

$$P\left[NPW_{\text{A}} \geq NPW_{\text{B}}\right]$$

If this is bigger than .5, Project A might be preferable to Project B (but again, common sense must prevail ... if Project A has a small but important probability of disaster, this criterion may not be a decisive one).