Stat 342 Example 1

Suppose that (x,y) has a jointly cont$^{\underline{s}}$ dsn (on $[0,1]^2$) with joint pdf

$$f(x,y) = (x+y)\, I\left[0 \le x \le 1 \text{ and } 0 \le y \le 1\right]$$
$$= \begin{cases} x+y & \text{if } (x,y) \in [0,1]^2 \\ 0 & \text{otherwise} \end{cases}$$

Then, for any $x \in [0,1]$

$$f(y|x) = \begin{cases} \dfrac{(x+y)}{\int_0^1 (x+y)\,dy} & \text{for } y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 2\left(\dfrac{x+y}{2x+1}\right) & \text{for } y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

So for $0 \le x \le 1$, $E[y|x] = \int y\, f(y|x)\,dy$

$$= \int_0^1 y\,(2)\left(\frac{x+y}{2x+1}\right) dy$$
$$\vdots$$
$$= \frac{1}{3}\left(\frac{3x+2}{2x+1}\right)$$

So, the optimal predictor of y based on x under SEL is

$$\hat{y}^{opt} = E[y|x] = \frac{1}{3}\left(\frac{3x+2}{2x+1}\right)$$

It's also possible to find an AEL optimal predictor of y based on x. For this one needs the median of a conditional dsn of $y|x$. To this end, consider the conditional cdf of $y|x$. This is (for $(x,y) \in [0,1]^2$)

$$F(y|x) = \int_0^y 2\left(\frac{x+t}{2x+1}\right)dt = \cdots = \left(\frac{2}{2x+1}\right)\left(\frac{y^2}{2} + xy\right)$$

a quadratic function of y. The conditional median of $y|x$, $\tilde{y}(x)$, solves

$$.5 = F(y|x) = \left(\frac{2}{2x+1}\right)\left(\frac{y^2}{2} + xy\right)$$

i.e. $\quad 0 = y^2 + 2xy - (x + \tfrac{1}{2})$

Use of the quadratic formula and the fact that $\tilde{y}$ must be positive to conclude that

$$\hat{y}^{opt}(x) = \tilde{y}(x) = -x + \sqrt{x^2 + (x+\tfrac{1}{2})^2}$$