Stat 342 Example 12

The $N(\mu,\sigma^2)$ distribution has pdf

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp - \frac{1}{2\sigma^2}(x-\mu)^2$$

and the fact is that its moment generating function is

$$M(t) = E\exp(tX)$$
$$= \int_{-\infty}^{\infty} \exp(tx) f(x)\,dx = \exp\left(\mu t + \tfrac{1}{2}\sigma^2 t^2\right)$$

Then note that if $V_1 \sim N(\mu_1, \sigma_1^2)$ independent of $V_2 \sim N(\mu_2, \sigma_2^2)$ the mgf of the sum i

$$M_{V_1+V_2}(t) = E\exp(t(V_1+V_2)) = E(\exp tV_1)(\exp tV_2)$$
$$\overset{\text{independence}}{=} E\exp(tV_1)\cdot E\exp(tV_2)$$
$$= \exp\left(\mu_1 t + \tfrac{1}{2}\sigma_1^2 t^2\right)\exp\left(\mu_2 t + \tfrac{1}{2}\sigma_2^2 t^2\right)$$
$$= \exp\left((\mu_1+\mu_2)t + \tfrac{1}{2}(\sigma_1^2+\sigma_2^2)t^2\right)$$

which is a $N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$ mgf. By the uniqueness of MGF's V_1+V_2 is thus $N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$