Stat 342 Example 14

The χ^2_1 pdf is (per Example 11)

$$f(x) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{x}} \exp - \frac{x}{2} \quad I[x>0]$$

As it turns out, the χ^2_1 mgf is

$$M(t) = \int_0^\infty \exp(tx) f(x)\,dx = (1-2t)^{-\frac{1}{2}}$$

Then if one defines the χ^2_ν dsn to be the dsn of ν independent χ^2_1 r.v.'s (squares of standard normals per Example 11), the χ^2_ν mgf must be νth power of this χ^2_1 mgf

$$M(t) = (1-2t)^{-\frac{\nu}{2}}$$

Then either using a definition of χ^2_ν as a sum of independent χ^2_1's or the uniqueness of moment generating functions, it's immediate that the sum of independent χ^2 r.v.'s is again χ^2 with degrees of freedom the sum of the component degrees of freedom. Further, since it's possible to argue that

$$f(x) = \frac{1}{2^{\nu/2}\Gamma(\frac{\nu}{2})} x^{\frac{\nu}{2}-1} \exp\left(-\frac{x}{2}\right) I[x>0]$$

is a pdf and that

$$\int_0^\infty \exp(tx) f(x)\,dx = (1-2t)^{-\frac{\nu}{2}}$$

it follows that $f(x)$ is the χ^2_ν pdf.