Stat 342 Example 15

For $x_1, x_2, \ldots, x_n$ iid $N(\mu, \sigma^2)$ an important quantity for statistical inference is

$$T = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}}$$

This can be rewritten slightly as

$$T = \frac{\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}}{\frac{s}{\sqrt{n}\,(\sigma/\sqrt{n})}} = \frac{Z}{\frac{s}{\sigma}} \qquad \text{for } Z = \frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}} \sim N(0,1)$$

Then, since $\frac{s}{\sigma} = \sqrt{\frac{s^2}{\sigma^2}} = \sqrt{\left(\frac{(n-1)s^2}{\sigma^2}\right)/(n-1)} = \sqrt{\frac{W}{n-1}}$

for $W = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$ independent of $\bar{x}$ (and thus Z), this T ratio has the form of "std normal over the root of an independent χ^2 variable over its degrees of freedom."

This $\frac{Z}{\sqrt{\frac{W}{\nu}}}$ "representational" definition of the t_ν distribution is the best way to introduce it in theoretical theorem-proving terms. It then can be shown that the corresponding (bell-shaped-flatter-than-standard-normal) pdf is

$$f(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

Probabilities follow from this pdf and are tabled or available from any decent statistical software package like R or JMP.

For example, the t_4 upper 5% pt (.95 quantile) is about 2.132. So based on a sample of size $n=5$ from a $N(\mu, \sigma^2)$ distribution

$$P\Big[\underbrace{-2.132 \le \frac{\bar{x}-\mu}{\frac{s}{\sqrt{5}}} \le 2.132}\Big] = .90$$

this event is exactly the event that

$$\bar{x} - 2.132\frac{s}{\sqrt{5}} \le \mu \le \bar{x} + 2.132\frac{s}{\sqrt{5}}$$

<u>So</u> for sample size $n=5$ we call

$$\bar{x} \pm 2.132\frac{s}{\sqrt{5}}$$

90% confidence limits for μ. That is, distributional facts for T give us an inference method for μ in the iid $N(\mu, \sigma^2)$ statistical model.

In general, the limits are

$$\bar{x} \pm t\frac{s}{\sqrt{n}}$$

a t_{n-1} % pt