Stat 342 Example 16

In a "two independent normal samples" model

$x_{11}, x_{12}, \ldots, x_{1n}$ are iid $N(\mu_1, \sigma_1^2)$ independent

$x_{21}, x_{22}, \cdots, x_{2n_2}$ that are iid $N(\mu_2, \sigma_2^2)$

An important quantity for elementary inference is then

$$F = \frac{s_1^2}{s_2^2} = \frac{\left(\frac{(n_1-1)s_1^2}{\sigma_1^2}\right)/(n_1-1)}{\left(\frac{(n_2-1)s_2^2}{\sigma_2^2}\right)/(n_2-1)} \cdot \frac{\sigma_1^2}{\sigma_2^2}$$

This then has the form

$$\frac{(W_1/\nu_1)}{(W_2/\nu_2)} \cdot \frac{\sigma_1^2}{\sigma_2^2}$$

for $W_1 = \frac{(n_1-1)s_1^2}{\sigma_1^2} \sim \chi^2_{n_1-1}$ independent of

$W_2 = \frac{(n_2-1)s_2^2}{\sigma_2^2} \sim \chi^2_{n_2-1}$

This motivates the definition of the F_{ν_1,ν_2} distribution as the distribution of

$$\frac{W_1/\nu_1}{W_2/\nu_2}$$

for $W_1 \sim \chi^2_{\nu_1}$ independent of $W_2 \sim \chi^2_{\nu_2}$. As it turns out, this definition leads to a pdf of the form

$$f(x) = \frac{1}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)} \left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} x^{\frac{\nu_1}{2}-1} \left(1 + \frac{\nu_1}{\nu_2}x\right)^{-\frac{\nu_1+\nu_2}{2}} I[x>0]$$

and probabilities are available by integrating with this form and are tabled and produced by statistical software.

Since $\frac{s_1^2}{s_2^2} = \left(\text{an } F_{n_1-1, n_2-1} \text{ r.v.}\right) \frac{\sigma_1^2}{\sigma_2^2}$

$$\frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2} \sim F_{n_1-1,\, n_2-1}$$

and one can find numbers # and ## so that (e.g.)

$$P\left[\# < \text{an } F_{n_1-1,\, n_2-1} \text{ random variable} < \#\#\right] = .95$$

i.e.

$$P\Bigg[\underbrace{\# < \frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2} < \#\#}_{\text{this event is exactly the event}}\Bigg] = .95$$

$$\frac{s_1}{s_2\sqrt{\#\#}} < \frac{\sigma_1}{\sigma_2} < \frac{s_1}{s_2\sqrt{\#}}$$

That is, a 95% confidence interval for $\frac{\sigma_1}{\sigma_2}$ is

$$\left(\frac{s_1}{s_2\sqrt{\#\#}},\ \frac{s_1}{s_2\sqrt{\#}}\right)$$