Stat 342 Example 2

As an example of 0-1 loss prediction/classification for $y \in \{0,1\}$ consider the following. Suppose

$$P[y=0] = .4 \quad \text{so that} \quad P[y=1] = .6$$

and that given y, x is normal $(y,1)$. This describes a joint dsn for (x,y) concentrated on $\mathbb{R} \times \{0,1\}$ (the pair of lines in $\mathbb{R}^2$ below).

The total probability on the top line .6 and given that $y=1$, $x \sim N(1,1)$. The total probability on the bottom line is .4 and given that $y=0$, $x \sim N(0,1)$. A joint "density" for (x,y) (that one adds across y and does Reiman integration across x) is

$$f(x,y) = \begin{cases} .4 \frac{1}{\sqrt{2\pi}} \exp -\frac{1}{2}(x-0)^2 & \text{if } y=0 \\ .6 \frac{1}{\sqrt{2\pi}} \exp -\frac{1}{2}(x-1)^2 & \text{if } y=1 \end{cases}$$

Then for any x

$$f(1|x) = P[y=1|x] \left(= \frac{f(x,y)}{f(x)} = \frac{f(x,y)}{f(x,0)+f(x,1)} \right)$$

$$= \frac{.6 \frac{1}{\sqrt{2\pi}} \exp -\frac{1}{2}(x-1)^2}{.6 \frac{1}{\sqrt{2\pi}} \exp -\frac{1}{2}(x-1)^2 + .4 \frac{1}{\sqrt{2\pi}} \exp -\frac{1}{2}(x-0)^2}$$

$$= \frac{\frac{3}{2} \exp -\frac{1}{2}(-1)(2x-1))}{\left(\frac{3}{2} \exp -\frac{1}{2}(-1)(2x-1)\right)+1}$$

This is increasing in x and

$$P[y=1 \mid x] = \frac{1}{2} \quad \text{when}$$

$$\exp\left(-\tfrac{1}{2}\right)(-1)(2x-1) = 1$$

i.e. $$x = \frac{1}{2} + \ln\frac{2}{3}$$

So, a 0-1 loss predictor/classifier decides in favor of $y=1$ when x exceeds $\frac{1}{2} + \ln\frac{2}{3}$. That is

$$\hat{y}^{opt}(x) = \mathbb{I}\left[x > \frac{1}{2} + \ln\frac{2}{3}\right]$$