Stat 342 Example 31

Suppose that $\underset{\sim}{x} = (x_1, \dots, x_n)$ has iid Poisson(λ) components x_i. Consider the statistic

$$T(\underset{\sim}{x}) = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Note that

$$E_\lambda T(\underset{\sim}{x}) = E_\lambda \bar{x} = \lambda$$

so that $T(\underset{\sim}{x}) = \bar{x}$ is unbiased for λ.

Now the Cramér-Rao lower bound for the variance of any unbiased estimator of λ is

$$\mathrm{Var}_\lambda \hat{\lambda}(\underset{\sim}{x}) \geq \frac{1}{I_{\underset{\sim}{x}}(\lambda)} = \frac{1}{n I_{x_1}(\lambda)}$$

$$= \frac{\lambda}{n}$$

But the variance of $T(\underset{\sim}{x}) = \bar{x}$ is $\frac{\lambda}{n}$. That is, $\bar{x}$ "achieves the C-R lower bound" and has (for every value of λ) the smallest possible variance among all unbiased estimators.

(For what it is worth, $\bar{x}$ here is sometimes called a "uniformly (in λ) minimum variance unbiased estimator" of λ.)