Stat 342 Example 33

Consider $\underset{\sim}{z} = (y_1, y_2, \cdots, y_n)$ with iid Exponential mean μ components y_i. The log-likelihood is

$$\ln f(\underset{\sim}{z}|\mu) = \ln\left(\prod_{i=1}^{n} \frac{1}{\mu} \exp\left(-\frac{y_i}{\mu}\right)\right)$$

$$= -n\ln\mu - \frac{1}{\mu}\sum y_i$$

So the "score function" (of μ) is

$$\frac{\partial}{\partial\mu} \ln f(\underset{\sim}{z}|\mu) = -\frac{n}{\mu} + \frac{1}{\mu^2}\sum y_i$$

and the "likelihood equation" is

$$\frac{\partial}{\partial\mu} \ln f(\underset{\sim}{z}|\mu) = 0$$

i.e.

$$-\frac{n}{\mu} + \frac{1}{\mu^2}\sum y_i = 0$$

which has the unique solution for μ of $\mu = \bar{y}$. $\bar{y}$ is in fact not only the only root of the likelihood equation, but also the unique maximizer of the likelihood, i.e.

$$\hat{\mu}_n^{MLE} = \bar{y}_n$$

Maximum likelihood theory OR the LLN promises that for any $\epsilon > 0$

$$P_{\mu_0}\left[\,|\hat{\mu}_n^{MLE} - \mu_0| > \epsilon\,\right] \longrightarrow 0$$

The distribution of $\hat{\mu}_n^{MLE}$ piles up at the "true value" of μ (the value under which probabilities are being calculated).

The FI in any y_i about μ can be gotten as follows.

$$\ln f(y|\mu) = -\ln\mu - \frac{y}{\mu}$$

$$\frac{\partial}{\partial\mu}\Big(\qquad\Big) = -\frac{1}{\mu} + \frac{y}{\mu^2}$$

$$\mathrm{Var}_\mu\Big(\qquad\Big) = \frac{1}{\mu^4}\mathrm{Var}_\mu(y) = \frac{1}{\mu^4}(\mu^2) = \frac{1}{\mu^2}$$

so maximum likelihood theory says that

$$(*)\qquad \frac{\hat{\mu}_n^{MLE} - \mu}{\sqrt{\frac{1}{nI_y(\mu)}}} = \frac{\bar{y}-\mu}{\sqrt{\frac{1}{n\left(\frac{1}{\mu^2}\right)}}} = \frac{\bar{y}-\mu}{\frac{\mu}{\sqrt{n}}}$$

is approximately standard normal. Since $\mu_y = \mu$ and $\sigma_y = \mu$ This is also

$$\frac{\bar{y}-\mu_y}{\sigma_y/\sqrt{n}}$$

and so this follows also from the CLT.

This means that limits

$$\bar{y} \pm z\frac{\mu}{\sqrt{n}}$$

are valid but unusable large n approximate confidence limits for μ. They can be "fixed" by fixing the use of $nI_y(\mu)$ in $(*)$. Fix #1 (the "expected information" fix) is to use instead $nI_y(\hat{\mu}_n^{MLE})$. This is $\frac{n}{\bar{y}^2}$ yielding limits

$$\bar{y} \pm z\frac{\bar{y}}{\sqrt{n}} \quad \text{i.e.} \quad \bar{y}\left(1 \pm z\frac{1}{\sqrt{n}}\right)$$

Fix # 2 (the "observed information" fix) is to replace $n I_y(\mu)$ with the negative curvature of the log-likelihood at the MLE. Then

$$-\frac{\partial^2}{\partial \mu^2} \ln f(\underset{\sim}{x} \mid \mu)\Big|_{\mu=\bar{y}} = \left(-\frac{n}{\mu^2} + \frac{2}{\mu^3}\sum y_i\right)\Big|_{\mu=\bar{y}}$$

$$= -\frac{n}{\bar{y}^2} + \frac{2}{\bar{y}^3}\sum y_i$$

$$= \frac{n}{\bar{y}^2}$$

So in this case, the 2nd fix works out to be exactly the same as the first.