Stat 342 Example 34

Suppose that $\underset{\sim}{y} = (y_1, y_2, \cdots, y_n)$ has iid components where a pdf for y is

$$f(y|\alpha) = \alpha y^{\alpha-1} I[0<y<1]$$

Then for $y \in (0,1)$

$$\frac{\partial}{\partial\alpha} \ln f(y|\alpha) = \frac{1}{\alpha} + \ln y \quad \text{and}$$

$$\frac{\partial^2}{\partial\alpha^2} \ln f(y|\alpha) = -\frac{1}{\alpha^2}$$

So the log-likelihood is

$$\ln f(\underset{\sim}{y}|\alpha) = \ln\left(\prod_{i=1}^{n} \alpha y_i^{\alpha-1}\right) = n \ln\alpha + (\alpha-1)\sum_{i=1}^{n} \ln y_i$$

and the likelihood equation is

$$\frac{\partial}{\partial\alpha} \ln f(\underset{\sim}{y}|\alpha) = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln y_i = 0$$

This has a unique solution $\alpha = -\dfrac{n}{\sum \ln y_i}$

That is also the unique maximizer of the log-likelihood, i.e.

$$\hat{\alpha}_n^{MLE} = -\frac{n}{\sum_{i=1}^{n} y_i}$$

Theory for the MLE says that $\hat{\alpha}_n^{MLE}$ is approximately normal with mean α and variance $\dfrac{1}{n I_y(\alpha)}$.

From above $\dfrac{\partial^2}{\partial^2\alpha} \ln f(y|\alpha) = -\dfrac{1}{\alpha^2}$ so that

$$I_y(\alpha) = -E_\alpha\left(-\frac{1}{\alpha^2}\right) = \frac{1}{\alpha^2}$$

and the variance of the approximating normal dsn is $\frac{\alpha^2}{n}$. This gives valid but useless large n approximate confidence limits for α of

$$\hat{\alpha}_n^{MLE} \pm z \frac{\alpha}{\sqrt{n}}$$

The first fix for this ("expected information" fix) is to replace $I_y(\alpha)$ with $I_y(\hat{\alpha}_n^{MLE})$ to give usable limits

$$\hat{\alpha}_n^{MLE} \pm z \frac{\hat{\alpha}_n^{MLE}}{\sqrt{n}}$$

The second fix is to replace $n\,I(\alpha)$ with the negative curvature of the log-likelihood at the MLE. This is

$$-\left(-\frac{n}{\alpha^2}\right)\Bigg|_{\hat{\alpha}_n^{MLE}} = \frac{n}{\left(\hat{\alpha}_n^{MLE}\right)^2}$$

In this particular case the 2 fixes work out to produce the same result.