Stat 342 Example 35

Suppose that $\underset{\sim}{x} = (y_1, y_2, \dots, y_n)$ has iid normal components with mean 0 and variance v. Then

$$f(y|v) = \frac{1}{\sqrt{2\pi v}} \exp - \frac{y^2}{2v}$$

$$\ln f(y|v) = -\frac{1}{2}\ln 2\pi - \frac{1}{2}\ln v - \frac{y^2}{2v}$$

$$\frac{\partial}{\partial v} \ln f(y|v) = -\frac{1}{2v} + \frac{y^2}{2v^2}$$

$$\frac{\partial^2}{\partial v^2} \ln f(y|v) = +\frac{1}{2v^2} - \frac{y^2}{v^3}$$

So $I_y(v) = -E_v\left(\frac{\partial^2}{\partial v^2} \ln f(y|v)\right) = -\frac{1}{2v^2} + \frac{v}{v^3} = \frac{1}{2v^2}$

To find an MLE for v, we note that

$$\ln f(\underset{\sim}{x}|v) = -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln v - \frac{1}{2v}\sum y_i^2$$

and $\frac{\partial}{\partial v} \ln f(\underset{\sim}{x}|v) = -\frac{n}{2v} + \frac{1}{v^2}\sum y_i^2$

The likelihood equation $\frac{\partial}{\partial v} \ln f(\underset{\sim}{x}|v) = 0$ has the unique root

$$v = \frac{1}{n}\sum_{i=1}^{n} y_i^2$$

which also maximizes the likelihood. That is,

$$\hat{v}_n^{MLE} = \frac{1}{n}\sum_{i=1}^{n} y_i^2$$

BTW, this also follows from the CLT ~~applied~~ to y_i^2's

Maximum likelihood theory implies that $\hat{v}_n^{MLE}$ is approximately normal with mean v and variance

$$\frac{1}{n I_y(v)} = \frac{1}{n\left(\frac{1}{2v^2}\right)} = \frac{2v^2}{n}$$

This produces valid but unusable large n approximate confidence limits for v

$$\hat{V}_n^{MLE} \pm z V \sqrt{\frac{2}{n}}$$

This can be "fixed" in the usual ways. The "expected information" fix replaces $n I_y(V)$ with $n I_y(\hat{V}_n^{MLE})$ ultimately producing

$$\hat{V}_n^{MLE} \left(1 \pm z \sqrt{\frac{2}{n}} \right)$$

The "observed information" fix replaces $n I_y(V)$ with the negative curvature of the log-likelihood at the MLE.

$$-\frac{\partial^2}{\partial V^2} \ln f(\underset{\sim}{x} | V) = -\frac{n}{2V^2} + \frac{1}{2V^3} \sum y_i^2$$

when this is evaluated at $V = \frac{1}{n} \sum y_i^2$ we get the same formula as for fix #1.

Here's an interesting complement to the above (not presented in class). In this example, earlier material can be used to get different, but nearly equivalent intervals for V. Definition 5 implies that as it is a sum of independent squared std normal variables,

$$\sum \left(\frac{x_i}{\sqrt{V}} \right)^2 = \frac{\sum x_i^2}{V} \sim \chi^2_n$$

So

$$P\left[\frac{1}{n}(\text{lower } 2.5\% \text{ pt of } \chi^2_n) < \frac{\hat{V}_n^{MLE}}{V} < \frac{1}{n}(\text{upper } 2.5\% \text{ pt of } \chi^2_n) \right] = .05$$

i.e.

$$** \quad P\left[\frac{\hat{V}_n^{MLE}}{\frac{1}{n}\chi^2_{upper}} < V < \frac{\hat{V}_n^{MLE}}{\frac{1}{n}\chi^2_{lower}} \right] = .05$$

But then, since for z_i iid $N(0,1)$ $\sum z_i^2 \sim \chi^2_n$

and the CLT says that

$\frac{1}{n}\sum z_i^2$ is approximately $N(1, \frac{2}{n})$

mean of z_1^2 — variance of z_1^2

or $\sum z_i^2$ is approximately $N(n, 2n)$. So

upper 2.5% pt of χ^2_n $\approx n + 1.96\sqrt{2n}$

lower 2.5% pt of χ^2_n $\approx n - 1.96\sqrt{2n}$

So (**) says that approximate 95% confidence limits for V are

$$\frac{\hat{V}_n^{MLE}}{\frac{1}{n}(n+1.96\sqrt{2n})} \quad \text{and} \quad \frac{\hat{V}_n^{MLE}}{\frac{1}{n}(n-1.96\sqrt{2n})}$$

i.e.

$$\frac{\hat{V}_n^{MLE}}{1+1.96\sqrt{\frac{2}{n}}} \quad \text{and} \quad \frac{\hat{V}_n^{MLE}}{1-1.96\sqrt{\frac{2}{n}}}$$

Then, since for small $|x|$

$$\frac{1}{1+x} \approx 1-x$$

these are approximately the same the limits derived from maximum likelihood theory.