Stat 342 Exam 2
Fall 2014

I have neither given nor received unauthorized assistance on this exam.

KEY

Name Signed **Date**

Name Printed

There are 10 questions on the following 5 pages. Do as many of them as you can in the available time. I will score each question out of 10 points AND TOTAL YOUR BEST 7 SCORES. (That is, this is a 70 point exam.)

1. Here are some facts about geometric distributions that you may use in what follows:

We'll say that $X \sim \text{Geo}(p)$ provided it has pmf

$$f(x \mid p) = \begin{cases} p(1-p)^{(x-1)} & \text{for } x = 1,2,3,4,\ldots \\ 0 & \text{otherwise} \end{cases}$$

For such a variable

$$\text{E}X = \frac{1}{p} \quad \text{and} \quad \text{Var}\, X = \frac{1-p}{p^2}$$

10 pts **a)** What is the Fisher information in a single observation X about the parameter p? (Give an expression involving p for this.)

$$\frac{\partial}{\partial p} \ln f(x \mid p) = \frac{\partial}{\partial p}\left(\ln p + (x-1)\ln(1-p) \right)$$

$$= \frac{1}{p} - (x-1)\frac{1}{1-p}$$

so $\text{Var}_p(\text{above}) = \text{Var}_p\left(\frac{1}{1-p}\right)x$

$$= \frac{1}{(1-p)^2}\text{Var}_p X = \frac{1}{(1-p)^2}\,\frac{(1-p)}{p^2}$$

$$= \frac{1}{(1-p)p^2}$$

10 pts **b)** Suppose that $X_1, X_2, \ldots, X_n$ are iid $\text{Geo}(p)$. Identify a 1-dimensional sufficient statistic for this model. Justify your choice (say why you know your statistic is sufficient).

$$f(\underset{\sim}{x} \mid p) = p^n (1-p)^{\Sigma(x_i - 1)}$$

$$= p^n (1-p)^{-n} (1-p)^{\Sigma x_i}$$

$$\Lambda_{\underset{\sim}{x}}\left(p, \tfrac{1}{2}\right) = \frac{p^n}{(1-p)^n}\,\frac{(1-p)^{\Sigma x_i}}{\left(\frac{1}{2}\right)^n}$$

which is a function of Σx_i. So Σx_i is sufficient.

10 pts **c)** Suppose that a sample of $n=10$ $\text{Geo}(p)$ variables produce the values below.

$$1,1,1,1,3,1,2,1,2,2$$

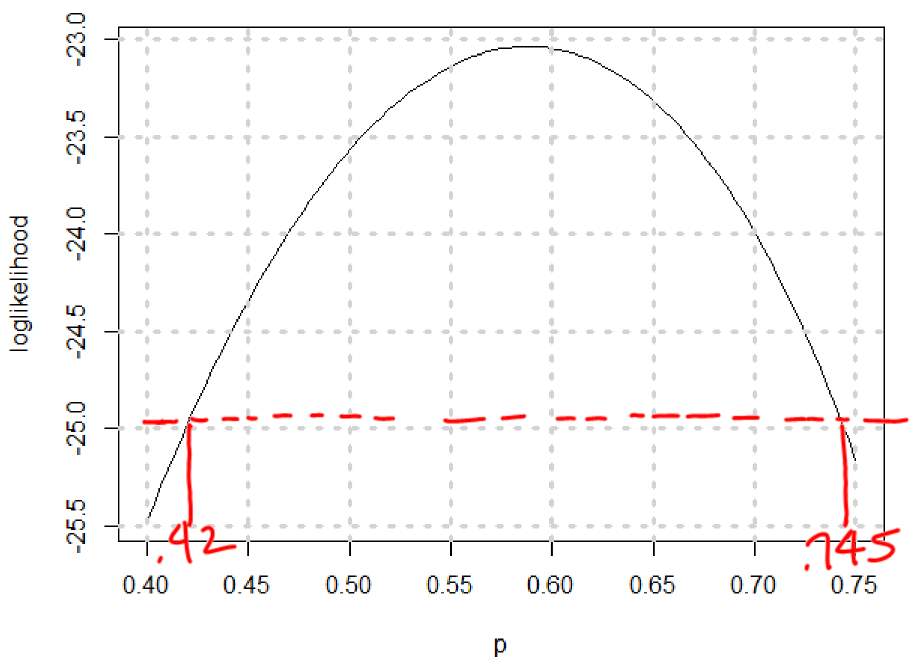
A maximum likelihood estimate of p based on this sample is $\frac{2}{3}$. Use this fact and find approximate 95% two-sided confidence limits for p.

$$I\left(\tfrac{2}{3}\right) = \frac{1}{\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)} = \frac{27}{4} \qquad nI\left(\tfrac{2}{3}\right) = \frac{270}{4}$$

So $\hat{p}^{MLE} \pm 1.96 \dfrac{1}{\sqrt{nI(\hat{p}^{MLE})}}$ i.e. $\left(\frac{2}{3}\right) \pm 1.96 \dfrac{1}{\sqrt{\frac{270}{4}}}$

(The interval based on observed information turns out to be the same as this one.)

10 pts **d)** Below is a plot of a geometric log-likelihood for a different sample of iid geometric observations. Use it and the fact that the upper 5% point of the χ^2_1 distribution is 3.841 to give approximate 95% confidence limits for this p. (The largest loglikelihood pictured below is about -23.035.) Indicate how you arrive at your limits.



$$-23.035 - \tfrac{1}{2}(3.841)$$
$$= -23.035 - 1.9205$$
$$= -24.955$$

2. Below is a table specifying pmfs for two possible discrete distributions for a random variable X. Call those $f(x|0)$ and $f(x|1)$. Use them in what follows.

	x									
	1	2	3	4	5	6	7	8	9	10
$f(x\|1)$	.08	.10	.05	.12	.08	.10	.10	.12	.05	.20
$f(x\|0)$	.04	.16	.10	.10	.08	.12	.14	.06	.10	.10

10 pts

a) Identify a minimal sufficient statistic T in the statistical model where observable $X \sim f(x|\theta)$ for $\theta \in \{0,1\}$. Provide values for $T(x)$ in the table below. What guarantees that your answer is minimal sufficient?

Compute $\Lambda_x(1,0) = \frac{f(x|1)}{f(x|0)}$

x	1	2	3	4	5	6	7	8	9	10
$T(x)$	2	$\frac{5}{8}$	$\frac{1}{2}$	1.2	1.0	$\frac{5}{6}$	$\frac{5}{7}$	2	.5	2

Rationale: The likelihood ratio statistic is minimal sufficient in a 2-distribution statistical model. (See Corollary 20)

10 pts

b) Consider a prior distribution for θ in the statistical model of part **a)** that has $g(0) = .6$ and $g(1) = .4$. Identify any test $a^{\text{opt}}(x)$ that has minimum (0-1 loss) Bayes risk in this context (find a Bayes optimal predictor for θ here). Provide values for $a^{\text{opt}}(x)$ in the table below.

x	1	2	3	4	5	6	7	8	9	10
$a^{\text{opt}}(x)$	1	0	0	0	0	0	0	1	0	1

We use the likelihood ratio statistic and set $a(x) = 1$ when $\Lambda_x(1,0) > \frac{g(0)}{g(1)} = \frac{.6}{.4} = 1.5$

10 pts **c)** For your test in **b)** evaluate Type I and Type II error probabilities.

Type I: $P_0[x=1 \text{ or } 9 \text{ or } 10] = .04 + .06 + .10 = .2$

Type II: $P_1[x \neq 1 \text{ or } 9 \text{ or } 10] = 1 - (.08 + .12 + .2)$

$= 1 - .4$

$= .6$

10 pts **3.** In a non-parametric bootstrap sample of size n from n distinct values $x_1, x_2, \ldots, x_n$, what is the probability that value x_1 does not occur in the bootstrap sample? This probability has a limit as $n \to \infty$. What is this limit?

$P[x_1 \text{ is not in the sample}] = \left(1 - \frac{1}{n}\right)^n \longrightarrow e^{-1}$

What then is the limit of

$$\mathrm{E}\frac{1}{n}\sum_{i=1}^{n} I\left[x_i \text{ is not in the bootstrap sample}\right]$$

(the expected fraction of values not appearing in the bootstrap sample) as $n \to \infty$?

$E\frac{1}{n}\sum I[\quad] = \frac{1}{n}\sum EI[\quad]$

$= \frac{1}{n}(n)\left(1 - \frac{1}{n}\right)^n$

$= \left(1 - \frac{1}{n}\right)^n \longrightarrow e^{-1}$

10 pts **4.** Suppose that $X \sim \text{Binomial}(p)$ and that

$$T(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases}$$

What is the Fisher information in $T(X)$ about p? (Note that $T(X) \sim \text{Binomial}\left(1-(1-p)^2\right)$.)

$$f(t|p) = (1-p)^{2(1-t)}\left(1-(1-p)^2\right)^t$$

$$\ln f(t|p) = 2\ln(1-p) + t\left[\ln\left(1-(1-p)^2\right) - 2\ln(1-p)\right]$$

$$\mathrm{Var}_p\left(\frac{\partial}{\partial p}\ln f(t|p)\right) = \left[\frac{\partial}{\partial p}\left[\ln\left(1-(1-p)^2\right) - 2\ln(1-p)\right]\right]^2 \mathrm{Var}_p T$$

$$\mathrm{Var}_p T = \left(1-(1-p)^2\right)(1-p)^2$$

How do you expect this to compare to the Fisher information in X about p? Why?

This will be smaller than $2p(1-p)$ because $T(X)$ is not sufficient for p. $X=1$ and $X=2$ have different likelihood functions, but the same value of $T(X)$.

10 pts **5.** Suppose that $X \sim \text{Poisson}(\lambda)$ and that what is of interest is (not λ itself, but rather) $\exp(-2\lambda)$. Show that the estimator

$$a(X) = \begin{cases} -1 & \text{if } X \text{ is odd} \\ 1 & \text{if } X \text{ is even} \end{cases}$$

is unbiased for $\exp(-2\lambda)$. (As it turns out, it is the *only* unbiased estimator, and therefore by default the *best* one!)

$$E_\lambda a(X) = \sum_{x=0}^{\infty} (-1)^x \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(-\lambda)^x}{x!} = e^{-\lambda}\left(e^{-\lambda}\right)$$

$$= e^{-2\lambda}$$

$a(X)$ is clearly silly. Identify a modification of it that you are sure will have better SEL risk function.

$e^{-2\lambda}$ is never negative. Changing the values of -1 to 0 is guaranteed to move the estimate closer to $e^{-2\lambda}$. So $a^*(X) = I[X \text{ is even}]$ has better risk function than $a(X)$.