## Percentile bootstrap confidence intervals

Suppose that a quantity $\theta = \eta(F)$ is of interest and that

$$T_n = \eta\,(\text{the empirical distribution of } \mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n)$$

Based on $B$ bootstrapped values $T^*_{n1}, T^*_{n2}, \ldots, T^*_{nB}$, define ordered values

$$T^*_{n(1)} \leq T^*_{n(2)} \leq \cdots \leq T^*_{n(B)}$$

Adopt the following convention (to locate lower and upper $\frac{\alpha}{2}$ points for the histogram/empirical distribution of the $B$ bootstrapped values). For

$$k_{\mathrm{L}} = \left\lfloor \frac{\alpha}{2}(B+1) \right\rfloor \text{ and } k_{\mathrm{U}} = (B+1) - k_{\mathrm{L}}$$

($\lfloor x \rfloor$ is the largest integer less than or equal to $x$) the interval

$$\left[T^*_{n(k_{\mathrm{L}})}, T^*_{n(k_{\mathrm{U}})}\right] \quad (1)$$

contains (roughly) the "middle $(1-\alpha)$ fraction of the histogram of bootstrapped values." This interval is called the (uncorrected) "$(1-\alpha)$ level bootstrap percentile confidence interval" for $\theta$.

The standard argument for why interval (1) might function as a confidence interval for $\theta$ is as follows. Suppose that there is an increasing function $m(\cdot)$ such that with

$$\phi = m(\theta) = m(\eta(F))$$

and

$$\widehat{\phi} = m(T_n) = m\,(\eta\,(\text{the empirical distribution of } \mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n))$$

for large $n$

$$\widehat{\phi} \dot\sim \mathrm{N}\left(\phi, w^2\right)$$

Then a confidence interval for $\phi$ is

$$\left[\widehat{\phi} - zw, \widehat{\phi} + zw\right]$$

and a corresponding confidence interval for $\theta$ is

$$\left[m^{-1}\left(\widehat{\phi} - zw\right), m^{-1}\left(\widehat{\phi} + zw\right)\right] \quad (2)$$

The argument is then that the bootstrap percentile interval (1) for large $n$ and large $B$ approximates this interval (2). The plausibility of an approximate correspondence between (1) and (2) might be argued as follows. Interval (2) is

$$\begin{aligned}
&\left[m^{-1}(\phi - zw), m^{-1}(\phi + zw)\right] \\
\approx\ &\left[m^{-1}\left(\text{lower } \frac{\alpha}{2} \text{ point of the dsn of } \widehat{\phi}\right), m^{-1}\left(\text{upper } \frac{\alpha}{2} \text{ point of the dsn of } \widehat{\phi}\right)\right] \\
=\ &\left[m^{-1}\left(m\left(\text{lower } \frac{\alpha}{2} \text{ point of } T_n \text{ dsn}\right)\right), m^{-1}\left(m\left(\text{lower } \frac{\alpha}{2} \text{ point of } T_n \text{ dsn}\right)\right)\right] \\
=\ &\left[\text{lower } \frac{\alpha}{2} \text{ point of the dsn of } T_n, \text{upper } \frac{\alpha}{2} \text{ point of the dsn of } T_n\right]
\end{aligned}$$

and one may hope that interval (1) approximates this last interval. The beauty of the bootstrap argument in this context is that one doesn't need to know the correct transformation $m$ (or the standard deviation $w$) in order to apply it.