Equivalence of the Full Model/Reduced Model and H_0:$\mathbf{C}\boldsymbol{\beta} = \mathbf{0}$ Sums of Squares

In a regression context, consider a (full rank model matrix)

$$\underset{n\times(r+1)}{\mathbf{X}} = (\mathbf{1}|\mathbf{x}_1|\mathbf{x}_2|\ldots|\mathbf{x}_r)$$

For $p < r$, let

$$\mathbf{X}_p = (\mathbf{1}|\mathbf{x}_1|\mathbf{x}_2|\ldots|\mathbf{x}_p)$$

and

$$\mathbf{U}_p = (\mathbf{x}_{p+1}|\mathbf{x}_{p+2}|\ldots|\mathbf{x}_r)$$

For

$$\mathbf{C} = \left(\underset{(r-p)\times(p+1)}{\mathbf{0}} \,|\, \underset{(r-p)\times(r-p)}{\mathbf{I}}\right)$$

consider

$$SS_{\mathrm{H}_0} = (\mathbf{C}\mathbf{b}_{\mathrm{OLS}})' \left(\mathbf{C}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{C}'\right)(\mathbf{C}\mathbf{b}_{\mathrm{OLS}})$$

which we invented for testing H_0:$\mathbf{C}\boldsymbol{\beta} = \mathbf{0}$. The object here is to show that

$$\begin{aligned} SS_{\mathrm{H}_0} &= SSE_{\mathrm{Reduced}} - SSE_{\mathrm{Full}} \\ &= \mathbf{Y}'\left(\mathbf{I} - \mathbf{P}_{\mathbf{X}_p}\right)\mathbf{Y} - \mathbf{Y}'\left(\mathbf{I} - \mathbf{P}_{\mathbf{X}}\right)\mathbf{Y} \\ &= \mathbf{Y}'\left(\mathbf{P}_{\mathbf{X}} - \mathbf{P}_{\mathbf{X}_p}\right)\mathbf{Y} \end{aligned}$$

To begin, note that with

$$\mathbf{A} = \left(\mathbf{U}_p'\left(\mathbf{I} - \mathbf{P}_{\mathbf{X}_p}\right)\mathbf{U}_p\right)^{-1}\mathbf{U}_p'\left(\mathbf{I} - \mathbf{P}_{\mathbf{X}_p}\right)$$

we have

$$\mathbf{C} = \mathbf{A}\mathbf{X}$$

Then,

$$\begin{aligned} SS_{\mathrm{H}_0} &= \left(\mathbf{A}\widehat{\mathbf{Y}}\right)'\left(\mathbf{A}\mathbf{P}_{\mathbf{X}}\mathbf{A}'\right)^{-1}\left(\mathbf{A}\widehat{\mathbf{Y}}\right) \\ &= \mathbf{Y}'\mathbf{P}_{\mathbf{X}}\mathbf{A}'\left(\mathbf{A}\mathbf{P}_{\mathbf{X}}\mathbf{P}_{\mathbf{X}}\mathbf{A}'\right)^{-1}\mathbf{A}\mathbf{P}_{\mathbf{X}}\mathbf{Y} \end{aligned}$$

Then write

$$\begin{aligned} \mathbf{P}_{\mathbf{X}}\mathbf{A}' &= \mathbf{P}_{\mathbf{X}}\left(\mathbf{I} - \mathbf{P}_{\mathbf{X}_p}\right)\mathbf{U}_p\left(\mathbf{U}_p'\left(\mathbf{I} - \mathbf{P}_{\mathbf{X}_p}\right)\mathbf{U}_p\right)^{-1} \\ &= \left(\mathbf{U}_p - \mathbf{P}_{\mathbf{X}_p}\mathbf{U}_p\right)\left(\mathbf{U}_p'\left(\mathbf{I} - \mathbf{P}_{\mathbf{X}_p}\right)\left(\mathbf{I} - \mathbf{P}_{\mathbf{X}_p}\right)\mathbf{U}_p\right)^{-1} \\ &= \left(\mathbf{U}_p - \mathbf{P}_{\mathbf{X}_p}\mathbf{U}_p\right)\left(\left(\mathbf{U}_p - \mathbf{P}_{\mathbf{X}_p}\mathbf{U}_p\right)'\left(\mathbf{U}_p - \mathbf{P}_{\mathbf{X}_p}\mathbf{U}_p\right)\right)^{-1} \end{aligned}$$

Use the abbreviation

$$\mathbf{U}_p^* = \mathbf{U}_p - \mathbf{P}_{\mathbf{X}_p}\mathbf{U}_p$$

and note that

$$\begin{aligned} SS_{\mathrm{H}_0} &= \mathbf{Y}'\mathbf{U}_p^*\left(\mathbf{U}_p^{*\prime}\mathbf{U}_p^*\right)^{-1}\left(\left(\mathbf{U}_p^{*\prime}\mathbf{U}_p^*\right)^{-1}\mathbf{U}_p^{*\prime}\mathbf{U}_p^*\left(\mathbf{U}_p^{*\prime}\mathbf{U}_p^*\right)^{-1}\right)^{-1}\left(\mathbf{U}_p^{*\prime}\mathbf{U}_p^*\right)^{-1}\mathbf{U}_p^{*\prime}\mathbf{Y} \\ &= \mathbf{Y}'\mathbf{P}_{\mathbf{U}_p^*}\mathbf{Y} \end{aligned}$$

It then suffices to show that $\mathbf{P}_{\mathbf{U}_p^*} = \mathbf{P}_{\mathbf{X}} - \mathbf{P}_{\mathbf{X}_p}$.

Christensen's Theorem B.47 shows that $\mathbf{P}_{\mathbf{X}} - \mathbf{P}_{\mathbf{X}_p}$ is a perpendicular projection matrix. His Theorem B.48 says that $C\left(\mathbf{P}_{\mathbf{X}} - \mathbf{P}_{\mathbf{X}_p}\right)$ is the "orthogonal complement of $C(\mathbf{X}_p)$ with respect to $C(\mathbf{X})$" defined on his page 395. That is,

$$C\left(\mathbf{P}_{\mathbf{X}} - \mathbf{P}_{\mathbf{X}_p}\right) = C(\mathbf{X}) \cap C(\mathbf{X}_p)^{\perp}$$

so it suffices to show that $C\left(\mathbf{U}_p^*\right) = C(\mathbf{X}) \cap C(\mathbf{X}_p)^{\perp}$.

Each column of $\mathbf{U}_p - \mathbf{P}_{\mathbf{X}_p}\mathbf{U}_p$ is a difference in a column of $\mathbf{U}_p$ and a linear combination of columns of $\mathbf{P}_{\mathbf{X}_p}$. Since $C(\mathbf{U}_p) \subset C(\mathbf{X})$ and $C\left(\mathbf{P}_{\mathbf{X}_p}\right) = C(\mathbf{X}_p) \subset C(\mathbf{X})$, each column of $\mathbf{U}_p - \mathbf{P}_{\mathbf{X}_p}\mathbf{U}_p$ is in $C(\mathbf{X})$ and $C\left(\mathbf{U}_p^*\right) \subset C(\mathbf{X})$.

So then note that if $\mathbf{v} \in C\left(\mathbf{U}_p^*\right)$ $\exists\, \boldsymbol{\gamma}$ such that

$$\mathbf{v} = \mathbf{U}_p^*\boldsymbol{\gamma}$$

For such a $\mathbf{v}$

$$\left(\mathbf{I} - \mathbf{P}_{\mathbf{X}_p}\right)\mathbf{v} = \left(\mathbf{I} - \mathbf{P}_{\mathbf{X}_p}\right)\mathbf{U}_p^*\boldsymbol{\gamma} = \left(\mathbf{I} - \mathbf{P}_{\mathbf{X}_p}\right)\left(\mathbf{I} - \mathbf{P}_{\mathbf{X}_p}\right)\mathbf{U}_p\boldsymbol{\gamma} = \left(\mathbf{I} - \mathbf{P}_{\mathbf{X}_p}\right)\mathbf{U}_p\boldsymbol{\gamma} = \mathbf{U}_p^*\boldsymbol{\gamma} = \mathbf{v}$$

so $\mathbf{v} \in C(\mathbf{X}_p)^{\perp}$. Thus $C\left(\mathbf{U}_p^*\right) \subset C(\mathbf{X}) \cap C(\mathbf{X}_p)^{\perp}$.

Finally, suppose that $\mathbf{v} \in C(\mathbf{X}) \cap C(\mathbf{X}_p)^{\perp}$ and for some

$$\boldsymbol{\gamma} = \begin{pmatrix} \underset{(p+1)\times 1}{\boldsymbol{\gamma}_1} \\ \underset{(r-p)\times 1}{\boldsymbol{\gamma}_2} \end{pmatrix}$$

write

$$\mathbf{v} = \mathbf{X}\boldsymbol{\gamma} = (\mathbf{X}_p|\mathbf{U}_p)\,\boldsymbol{\gamma} = \left(\mathbf{X}_p|\mathbf{U}_p^* + \mathbf{P}_{\mathbf{X}_p}\mathbf{U}_p\right)\boldsymbol{\gamma} = \mathbf{X}_p\boldsymbol{\gamma}_1 + \left(\mathbf{U}_p^* + \mathbf{P}_{\mathbf{X}_p}\mathbf{U}_p\right)\boldsymbol{\gamma}_2$$

Then continuing

$$\mathbf{v} = \mathbf{X}_p\boldsymbol{\gamma}_1 + \mathbf{P}_{\mathbf{X}_p}\mathbf{U}_p + \mathbf{U}_p^*\boldsymbol{\gamma}_2 = \mathbf{P}_{\mathbf{X}_p}\mathbf{X}_p\boldsymbol{\gamma}_1 + \mathbf{P}_{\mathbf{X}_p}\mathbf{U}_p + \mathbf{U}_p^*\boldsymbol{\gamma}_2$$

so

$$\mathbf{v} = \mathbf{P}_{\mathbf{X}_p}(\mathbf{X}_p|\mathbf{U}_p)\,\boldsymbol{\gamma} + \mathbf{U}_p^*\boldsymbol{\gamma}_2 = \mathbf{P}_{\mathbf{X}_p}\mathbf{v} + \mathbf{U}_p^*\boldsymbol{\gamma}_2$$

Since $\mathbf{v} \perp C(\mathbf{X}_p)$ we then have $\mathbf{v} = \mathbf{U}_p^*\boldsymbol{\gamma}_2$ and thus $\mathbf{v} \in C\left(\mathbf{U}_p^*\right)$. So $C(\mathbf{X}) \cap C(\mathbf{X}_p) \subset C\left(\mathbf{U}_p^*\right)$ and we're done.