This question concerns a model and inference using that model for so-called "circular data." These are data that take values in the interval $[-\pi,\pi)$ (and can thus be identified with points on the unit circle). A useful standard model for circular data is the von Mises distribution with "direction parameter" $\mu\in[-\pi,\pi)$ and "concentration parameter" $\kappa\geq 0$. This distribution has probability density

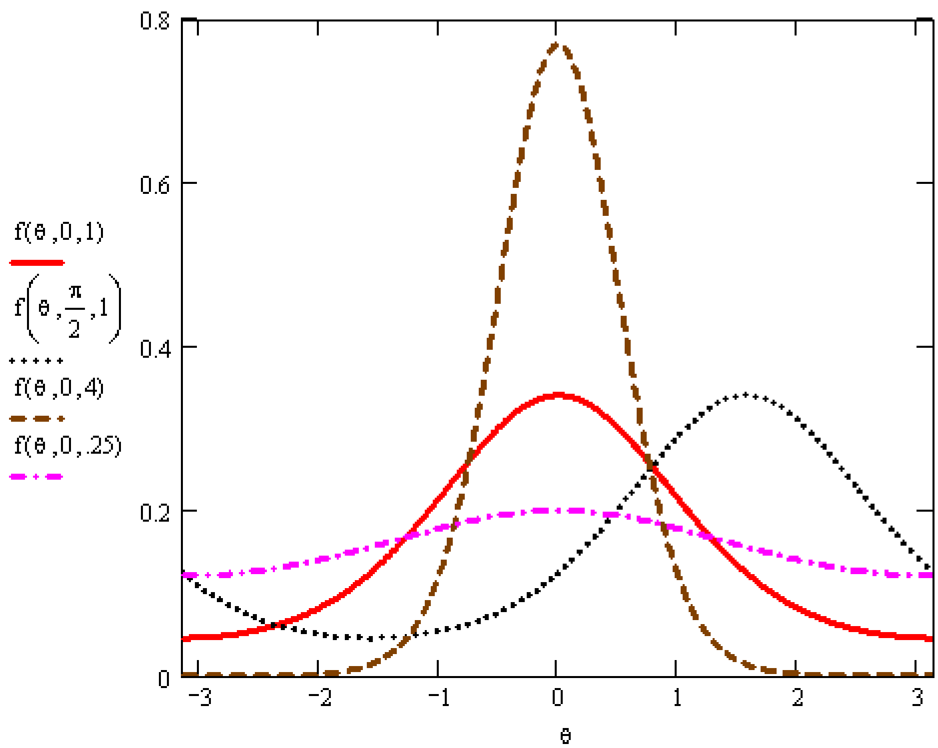
$$f(\theta\,|\,\mu,\kappa)=C(\kappa)\exp(\kappa\cos(\theta-\mu))I[-\pi\leq\theta<\pi]$$

for normalizing constant

$$C(\kappa)=\left(\int_{-\pi}^{\pi}\exp(\kappa\cos\theta)d\theta\right)^{-1}$$

Plots of four von Mises probability densities are below in Figure 1.

Figure 1: Plots of Four Different von Mises Probability Densities



1) Carefully define convergence in distribution. (That is, for random variables X_n with cdfs F_n and X with cdf F, what does it mean for X_n to converge to X in distribution?) Then suppose that θ_n has the von Mises distribution with direction $\mu=0$ and concentration parameter $\kappa=\frac{1}{n}$. Argue carefully using your definition that the sequence $\{\theta_n\}$ converges in distribution. (You may use without proof the fact that if $\{g_n\}$ is a sequence of continuous functions on the finite closed interval $[a,b]$ converging to a function g and there is a positive constant A with $0\leq g_n(t)\leq A$ for all n and $t\in[a,b]$, then $\int_a^b g_n(t)\,dt\to\int_a^b g(t)\,dt$.)

2) For Y a random variable with pdf $h(y)$ on $\Re$ and positive constant c, what is the pdf of the random variable cY? Use your answer in the following. Suppose that θ_n has the von Mises distribution with direction parameter $\mu = 0$ and concentration parameter $\kappa = n$ and let $Z_n = \sqrt{n}\,\theta_n$. Argue that the pdf of Z_n, say g_n, approximates the standard normal pdf as $n \to \infty$. (Hint: It suffices to show that for large n, $\ln g_n(z) \approx d(n) - \frac{z^2}{2}$ for some $d(n)$ depending only upon n and not z. Consider Taylor's theorem.)

3) Completely describe an algorithm that you could use to simulate from the von Mises distribution with direction parameter $\mu = \pi/2$ and concentration parameter $\kappa = 1$. (Just naming such an algorithm is not sufficient. You must say exactly what needs computing and how to compute it.)

Suppose that $\theta_1, \theta_2, \ldots, \theta_n$ are iid with the von Mises distribution with direction parameter $\mu = 0$ and unknown concentration parameter $\kappa > 0$.

4) Find a minimal sufficient statistic for the parameter κ and carefully argue that it is minimal sufficient. Then argue carefully that the family of von Mises distributions with $\mu = 0$ has monotone likelihood ratio in the statistic that you identify.

5) Describe an optimal size $\alpha = .05$ test of $\text{H}_0 : \kappa \geq 1$ versus $\text{H}_\text{a} : \kappa < 1$ based on $\theta_1, \theta_2, \ldots, \theta_n$. Describe how you would find the appropriate cut-off value for your test using simulation. Then show how you could use numerical integration to evaluate some integrals and then a large sample normal approximation to *approximate* the cut-off value for your test.

Now consider the problem of inference for *both* κ and μ based on a sample of, say, $n = 20$ iid von Mises observations. In fact, suppose that one observes

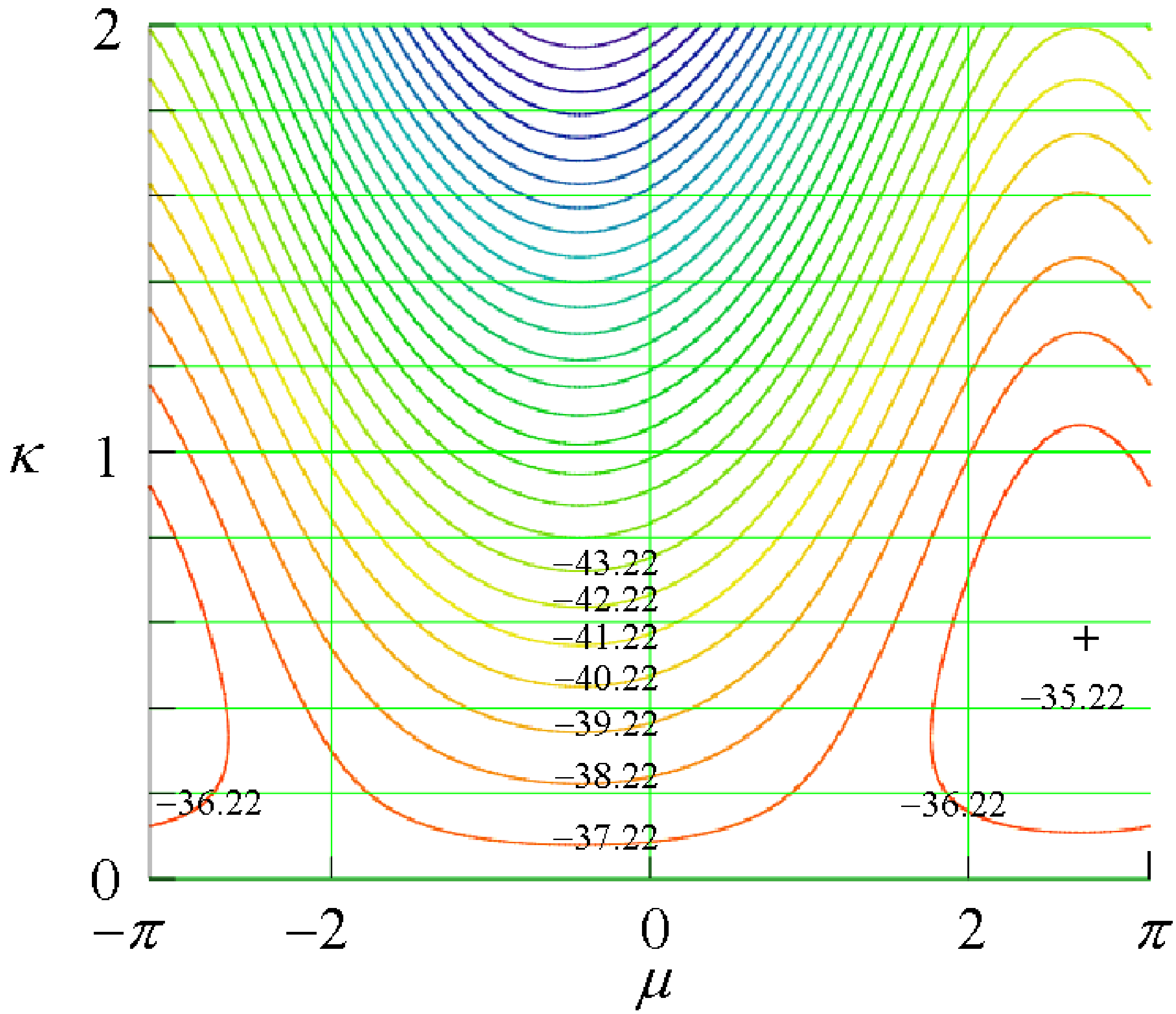
$$-3.135, -1.292, 1.605, .534, 2.405, -1.593, 2.026, 0.174, -2.542, -1.965,$$
$$3.083, -2.286, -3.094, 1.412, 1.663, -1.709, 1.080, -2.802, 2.269, 1.367$$

Figure 2 on the next page is a contour plot of the loglikelihood for this data set. This function has a maximum of about -35.22 at $(\mu, \kappa) = (2.701, .57)$. The matrix of 2nd partial derivatives of the loglikelihood at $(2.701, .57)$ is approximately

$$\begin{pmatrix} \frac{\partial^2}{\partial \mu^2} \text{loglikelihood} & \frac{\partial^2}{\partial \mu \partial \kappa} \text{loglikelihood} \\ \frac{\partial^2}{\partial \mu \partial \kappa} \text{loglikelihood} & \frac{\partial^2}{\partial \kappa^2} \text{loglikelihood} \end{pmatrix} = \begin{pmatrix} -3.128 & 0 \\ 0 & -8.882 \end{pmatrix}$$

6) Exactly what function of μ and κ has been plotted in Figure 2? Give a formula. (You may abbreviate the observed values -3.135 through 1.367 as $\theta_1, \theta_2, \ldots, \theta_{20}$.)

Figure 2: A Contour Plot of a Particular von Mises LogLikelihood Function



7) Consider the hypothesis $H_0 : (\mu, \kappa) = (0, .8)$. Does a likelihood ratio test of this hypothesis versus H_a : not H_0 (using a large sample approximation for the null distribution) reject this hypothesis at level $\alpha = .01$? Explain carefully.

8) What are a sensible point estimate for μ and a corresponding standard error for the estimate based on this likelihood function? In light of the fact that $\mu \in [-\pi, \pi)$ do these lead to an appropriate large sample confidence set for μ? Explain.

9) Identify appropriate (individual) approximate 99% confidence sets for both μ and κ based on profile loglikelihood functions corresponding to Figure 2. (Give numerical descriptions of these sets.)