a) Prove the following simple lemma. (You may use the lemma in what follows even if you can not prove it.).

__**Lemma**__ Suppose that F is a continuous distribution with probability density function on $(-\infty,\infty)$

$$f(x) = C\exp\left(ax^2 + bx\right)$$

(for real numbers $a < 0, C,$ and b). Then F is Normal with mean $\mu = -b/2a$ and variance $\sigma^2 = -1/2a$.

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b) Suppose that (T,W) is a random vector such that $T \sim \mathrm{N}\left(\delta,\gamma^2\right)$ and that conditioned on $T = t$, $U \sim \mathrm{N}\left(t,\eta^2\right)$. Find the conditional distribution of T given that $U = u$.

c) Now suppose that $\mu \sim \mathrm{N}\left(0,\gamma^2\right)$ and that conditioned on μ, variables $W_1, W_2, \ldots, W_n$ are iid $\mathrm{N}(\mu,1)$. Let $\overline{W_n} = \frac{1}{n}\sum_{i=1}^{n} W_i$. What is the conditional distribution of μ given that $\overline{W_n} = \overline{w}_n$? (Hint: what is the conditional distribution of $\overline{W_n}$ given μ?) Evaluate the function of w,

$$m_n(w) \equiv \mathrm{E}\left[\mu \mid \overline{W_n} = w\right]$$

d) Now suppose that μ is a fixed unknown quantity and that the variables $W_1, W_2, \ldots, W_n$ are iid $\mathrm{N}(\mu,1)$. With $m_n(w)$ as defined in **c)** consider the random quantity $m_n\left(\overline{W_n}\right)$. Show that this converges to a constant in probability and identify the limit.

Now suppose that n values $0 = x_1 < x_2 < \cdots < x_n = 1$ are known, and that for two real numbers μ_1 and μ_2 and a $c \in (0,1)$ we define the function

$$\mu(x) \equiv \mu_1 I[x < c] + \mu_2 I[x \ge c]$$

Suppose further that (given the parameters $\mu_1, \mu_2,$ and c) variables $Y_1, Y_2, \ldots, Y_n$ are independent Normal random variables with variance 1, and means

$$\mathrm{E}\,Y_i = \mu_i = \mu(x_i)$$

(Y_i has mean μ_1 if $x_i < c$, and otherwise has mean μ_2). Consider the statistical problem with parameter vector (μ_1, μ_2, c).

e) Write out a likelihood for this problem, $L_n(\mu_1, \mu_2, c)$. For fixed c, what values of μ_1 and μ_2 maximize $L_n(\cdot,\cdot,c)$? Call these $\hat{\mu}_1(c)$ and $\hat{\mu}_2(c)$ and use the notations

$$f(y \mid \mu) \text{ for the } \mathrm{N}(\mu,1) \text{ pdf} \quad \text{and} \quad n_1(c) = \sum_{i=1}^{n} I[x_i < c]$$

f) Is there a unique maximum likelihood estimator for the parameter vector (μ_1, μ_2, c)? Explain carefully.

g) As explicitly as is possible, give a likelihood ratio test statistic for testing the hypothesis $\mathrm{H_0}: c = .5$ versus $\mathrm{H_a}: c \neq .5$.

Consider a Bayes version of the inference problem for (μ_1, μ_2, c). In particular, suppose that we give (μ_1, μ_2, c) a (prior) distribution G under which the parameters are independent with

$$\mu_1 \sim \mathrm{N}(0, \gamma^2)$$
$$\mu_2 \sim \mathrm{N}(0, \gamma^2)$$
$$c \sim U(0,1)$$

Let $g(\mu \mid 0, \gamma^2)$ stand for the $\mathrm{N}(0, \gamma^2)$ probability density and use the notation

$$h_1(c, Y_1, Y_2, \ldots, Y_n) = \int_{-\infty}^{\infty} \left(\prod_{i=1}^{n_1(c)} f(Y_i \mid \mu) \right) g(\mu \mid 0, \gamma^2)\, d\mu \quad \text{and}$$

$$h_2(c, Y_1, Y_2, \ldots, Y_n) = \int_{-\infty}^{\infty} \left(\prod_{i=n_1(c)+1}^{n} f(Y_i \mid \mu) \right) g(\mu \mid 0, \gamma^2)\, d\mu$$

h) Evaluate $\mathrm{E}[\mu_1 \mid c, Y_1, Y_2, \ldots, Y_n]$

i) Write (in terms of the functions h_1 and h_2) a conditional pdf for c given $Y_1, Y_2, \ldots, Y_n$. (Notice that this pdf is constant on each interval (x_{i-1}, x_i).)

j) Use your answers to **h)** and **i)** to evaluate $\mathrm{E}[\mu_1 \mid Y_1, Y_2, \ldots, Y_n]$

k) For an integer $1 < i < n$ evaluate $\mathrm{E}[\mu(x_i) \mid Y_1, Y_2, \ldots, Y_n]$