Stat 542-543 II

Throughout this question, for $\theta_1 > 0$ and $\theta_2 \in \mathcal{R},$ let $f(x|\theta_1, \theta_2)$ be the two parameter exponential probability density

$$f(x|\theta_1,\theta_2) = \begin{cases} \theta_1 \exp(-\theta_1(x-\theta_2)) & \text{for } x \geq \theta_2 \\ 0 & \text{otherwise} \end{cases} .$$

(a) Show that if $U \sim \text{Uniform}(0,1)$, then $Y = -\ln U$ has the "standard" exponential distribution (i.e. the exponential distribution with $\theta_1 = 1$ and $\theta_2 = 0$).

(b) Based on the fact in (a) and the availability of a uniform random number generator, how do you suggest simulating a random sample of size n from the Exponential(θ_1, θ_2) distribution? (Here we suppose that θ_1 and θ_2 are known constants.)

Let $X_1, X_2, ..., X_n$ be iid with marginal density $f(x|\theta_1, \theta_2)$.

(c) Find the (joint) maximum likelihood estimators of θ_1 and θ_2 based on $X_1, X_2, ..., X_n$.

(d) Argue carefully that your MLE of θ_1 from (c) is consistent.

(e) Give the form of the likelihood ratio tests of
i) $H_0: \theta_1 = \theta_1^0$ vs H_a: not H_0
and ii) $H_0: \theta_2 = \theta_2^0$ vs H_a: not H_0
(Write complete formulas for the test statistics and indicate what kinds of values of these will cause rejection of the null hypotheses. But you need NOT simplify the forms of your statistics nor speculate how to choose cut-offs to get α level tests.)

For each $i = 1, 2, ..., n$ let

$$X_i^* = X_i \text{ rounded to the nearest integer}$$

and now suppose that one gets to observe not the X_i's but rather the X_i^*'s. In fact, suppose that the frequency distribution of what is observed in a sample of $n = 20$ is

x^*	5	6	7	8	9
frequency	7	8	2	2	1

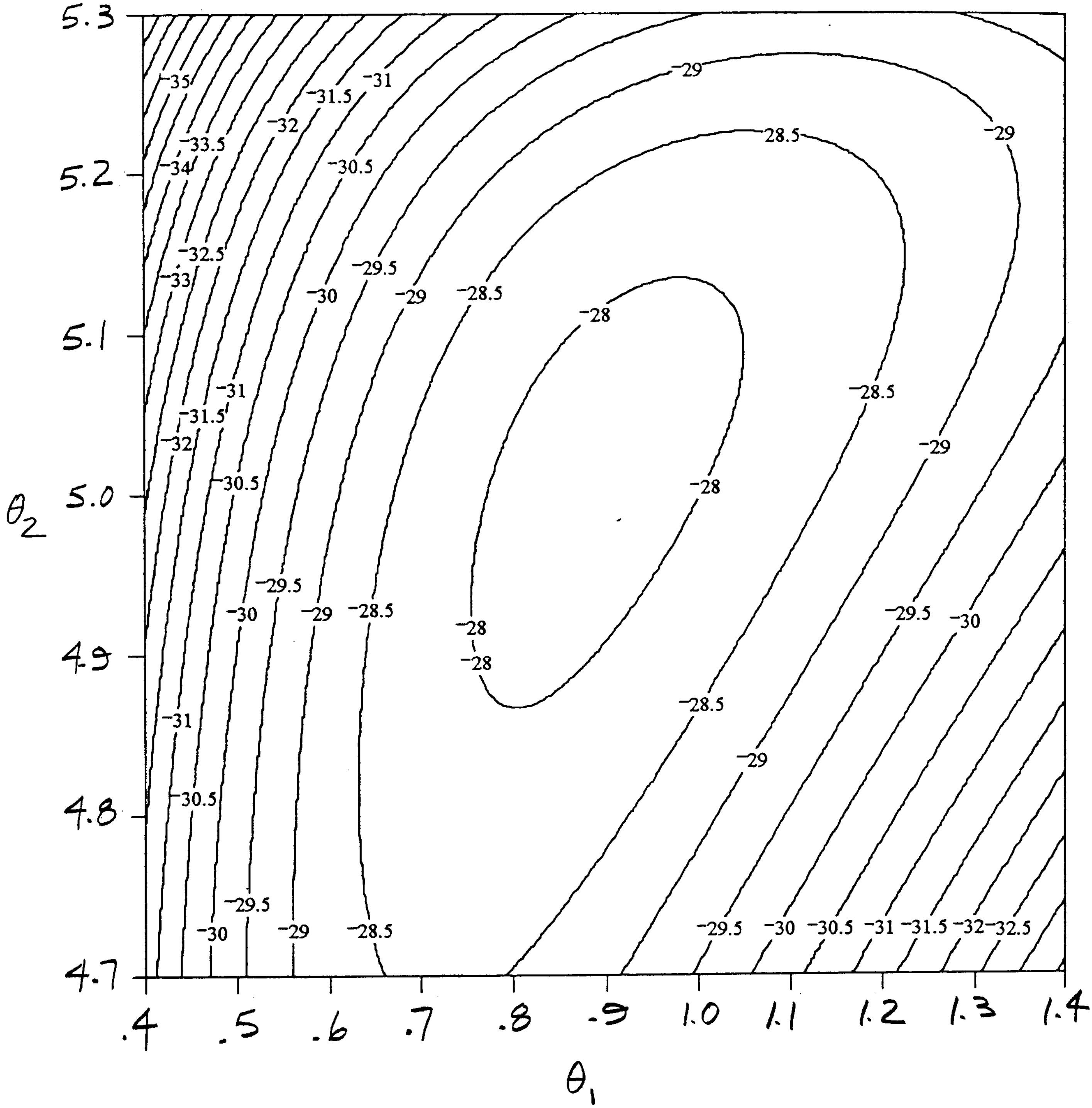
(f) Write out an appropriate likelihood function $L^*(\theta_1, \theta_2)$ based on these data. (Hint: For $x^* = 5, 6, 7, 8$ and 9, what are $P_{\theta_1,\theta_2}[X^* = x^*]$? Be careful. You will need one prescription for the likelihood in cases where $\theta_2 < 4.5$ and another for cases where $\theta_2 \geq 4.5$.)

As a matter of fact, the likelihood function referred to in (f) is maximized at $\theta_1 = .894$ and $\theta_2 = 5.018$. With $\mathcal{L}(\theta_1, \theta_2) = \ln L^*(\theta_1, \theta_2)$, $\mathcal{L}(.894, 5.018) = -27.833$ and second partial derivatives of $\mathcal{L}$ evaluated at $(.894, 5.018)$ are

$$\frac{\partial^2 \mathcal{L}}{\partial \theta_1^2} = -23.9, \ \frac{\partial^2 \mathcal{L}}{\partial \theta_2^2} = -29.7 \text{ and } \frac{\partial^2 \mathcal{L}}{\partial \theta_1 \partial \theta_2} = 16.0 \ .$$

(g) Give maximum likelihood estimates of θ_1 and θ_2 and appropriate standard errors for the estimates. Do the data suggest that maximum likelihood estimators based on X_i^*'s will be positively correlated or negatively correlated? Explain.

(h) Attached to this question is a contour plot of $\mathcal{L}(\theta_1, \theta_2)$. Use it and test the hypothesis H_0:$\theta_1 = 1.30$ and $\theta_2 = 4.90$ vs H_a:not H_0, with $\alpha \approx .05$. (There is a table of χ^2 percentage points also attached to this question.)



5.3
5.2
5.1
5.0
4.9
4.8
4.7
θ_2
.4
.5
.6
.7
.8
.9
1.0
1.1
1.2
1.3
1.4
θ_1
-35
-34
-33.5
-33
-32.5
-32
-31.5
-31
-30.5
-30
-29.5
-29
-28.5
-28