Stat 543 I

Suppose first that X is a discrete random variable with probability mass function $f(x|\theta)$ satisfying standard regularity conditions for θ in Θ an open interval in $\mathcal{R}$.

(a) **Define** $I_X(\theta_0)$, the Fisher information in X about the parameter θ, evaluated at $\theta_0 \in \Theta$.

(b) Suppose that for another open interval Γ in $\mathcal{R}$, the differentiable increasing function h maps Γ onto Θ. Consider the (reparameterized) model for X defined by

$$g(x|\gamma) \doteq f(x|h(\gamma)) \text{ for } \gamma \in \Gamma.$$

Express the Fisher information about the parameter γ, evaluated at $\gamma_0 \in \Gamma$, in terms of the functions $I_X(\theta)$ and $h(\gamma)$. (Write h' for the derivative of h.) **Argue** carefully that your expression is correct.

Suppose now that $Y_1, Y_2, ..., Y_n$ are independent Bernoulli random variables. For known constants $t_1, t_2, ..., t_n$ and unknown parameters $\alpha \in \mathcal{R}$ and $\beta \in \mathcal{R}$ we will suppose that

$$P[Y_i = 1] = \frac{1}{1 + \exp(-(\alpha + \beta t_i))} \quad . \tag{1}$$

This is equivalent to assuming that

$$\ln\left(\frac{P[Y_i = 1]}{1 - P[Y_i = 1]}\right) = \alpha + \beta t_i \quad . \tag{2}$$

(c) **Write out** the log-likelihood function, $\mathcal{L}(\alpha, \beta)$, for this problem **and** a pair of equations that will have to be solved in order to find maximum likelihood estimates of the parameters α and β.

For a particular famous data set with $n = 23$, two figures summarizing a likelihood analysis of this problem are attached to this question. Use them in what follows. It will also be helpful to know that MLE's for this data set are $\widehat{\alpha} = 15.044$ and $\widehat{\beta} = -.2322$ and $\mathcal{L}(15.044, -.2322) = -10.158$.

For $\beta \in \mathcal{R}$, let $\alpha^*(\beta)$ be a maximizer of $\mathcal{L}(\alpha, \beta)$ considered as a function of α. Define

$$\mathcal{L}^*(\beta) = \mathcal{L}(\alpha^*(\beta), \beta) \quad .$$

Figure 1 is a plot of this function.

(d) Note that $\beta = 0$ in either (1) or (2) is the case where $P[Y = 1]$ doesn't depend upon t. **Is** $\mathrm{H}_0 : \beta = 0$ a plausible hypothesis in the light of the Figure 1? **Explain**. (Hint: Consider a likelihood ratio test of H_0.)

(e) **Give** an approximate 90% confidence interval for β based on Figure 1.

In the real problem, the quantity $p_{31} \doteq \frac{1}{1+\exp(-(\alpha+31\beta))}$ was of vital interest. (This is the "success probability" for a Y with $t = 31$.) It is possible to parameterize this problem in terms of p_{31} and, say, $p_{72} \doteq \frac{1}{1+\exp(-(\alpha+72\beta))}$ (the "success probability" for a Y with $t = 72$.)

(f) **Give** MLE's of p_{31} and p_{72} and argue carefully that your values are correct.

(g) The hope was that $p_{31}$was in fact small. Figure 2 is a plot outlining the region in (p_{31}, p_{72})-space with log-likelihood exceeding -12.4605. Carefully **interpret** Figure 2 in the light of the hope about p_{31}.

Figure 1

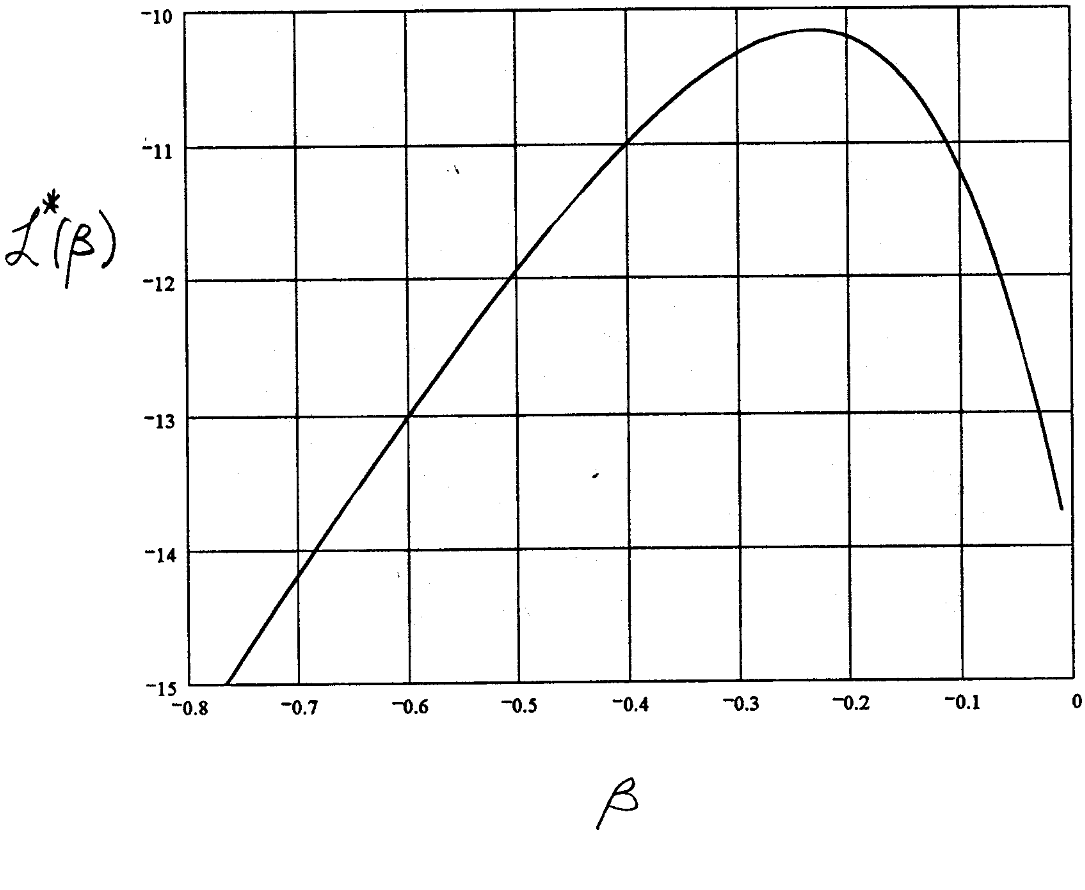

Figure 2