Stat 543 I

Suppose that for $\alpha \in [0,1]$, X is a random variable with probability density

$$f(x|\alpha) = \alpha f_1(x) + (1-\alpha) f_0(x) \qquad (*)$$

where $f_0(x)$ is the $\mathrm{N}(0,1)$ density and $f_1(x)$ is the $\mathrm{N}(1,1)$ density.

a) Find the mean and variance of X, $\mathrm{E}_\alpha X$ and $\mathrm{Var}_\alpha X$.

b) Show that the maximum likelihood estimator of α based on the single observation, X, is

$$\widehat{\alpha} = \begin{cases} 1 & \text{if } X > .5 \\ 0 & \text{otherwise} \end{cases} .$$

Compute the mean and variance of this estimator. Is $\widehat{\alpha}$ unbiased for α?

c) Argue that the mean squared error of $\widehat{\alpha}$ as an estimator of α is no more than $.25 + (.3085)^2$. Use this fact and compare $\widehat{\alpha}$ and X in terms of mean squared error.

d) Set up, but do not try to evaluate an integral giving $I(\alpha)$ the Fisher information in X concerning $\alpha \in (0,1)$.

Now consider estimation of α based on a sample $X_1, X_2, \ldots, X_n$ that are iid with density $(*)$. Let $\widehat{\alpha}_n$ be the MLE of α based on the n observations and let $\overline{X}_n$ be the usual sample mean.

e) Attached to this question are several plots. Figure 1 gives a plot of $I(\alpha)$ from d). Figure 2 gives a plot of both $1/I(\alpha)$ and $1+\alpha-\alpha^2$. Make use of these and compare the large sample distributions of $\widehat{\alpha}_n$ and $\overline{X}_n$. On the basis of large sample considerations, which of these is the better estimator of α? Explain carefully.

A particular sample of $n = 20$ observations produced the loglikelihood function plotted in Figure 3.

f) What, approximately, is the maximum likelihood estimate of α for this sample? (Use Figure 3.)

g) Use Figure 3 and give an approximate 90% confidence interval for α. (The largest loglikelihood plotted in Figure 3 is about -31.636.)

Figure 1

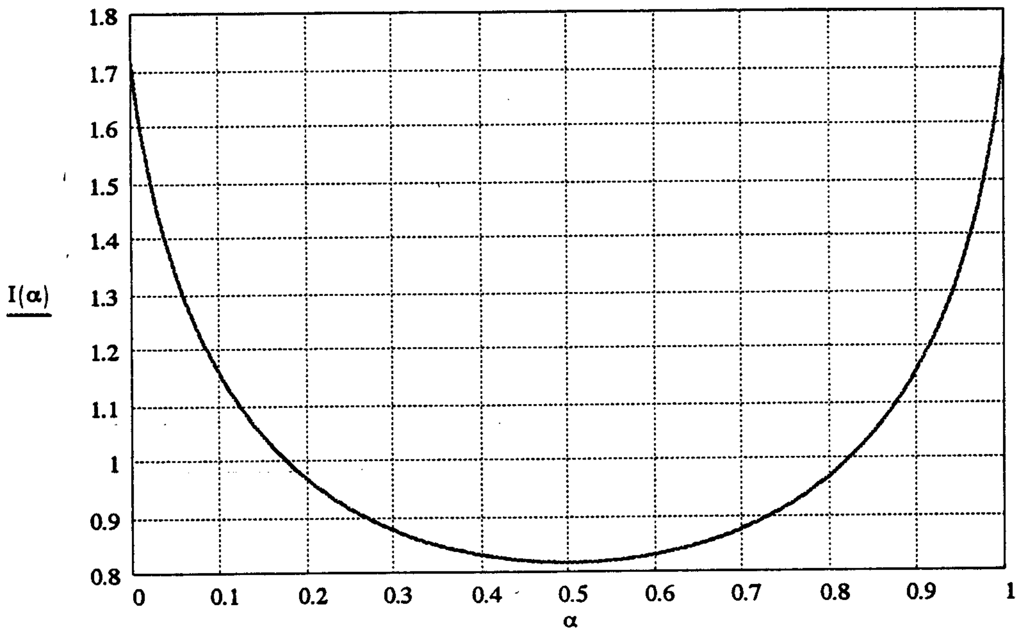


Figure 2

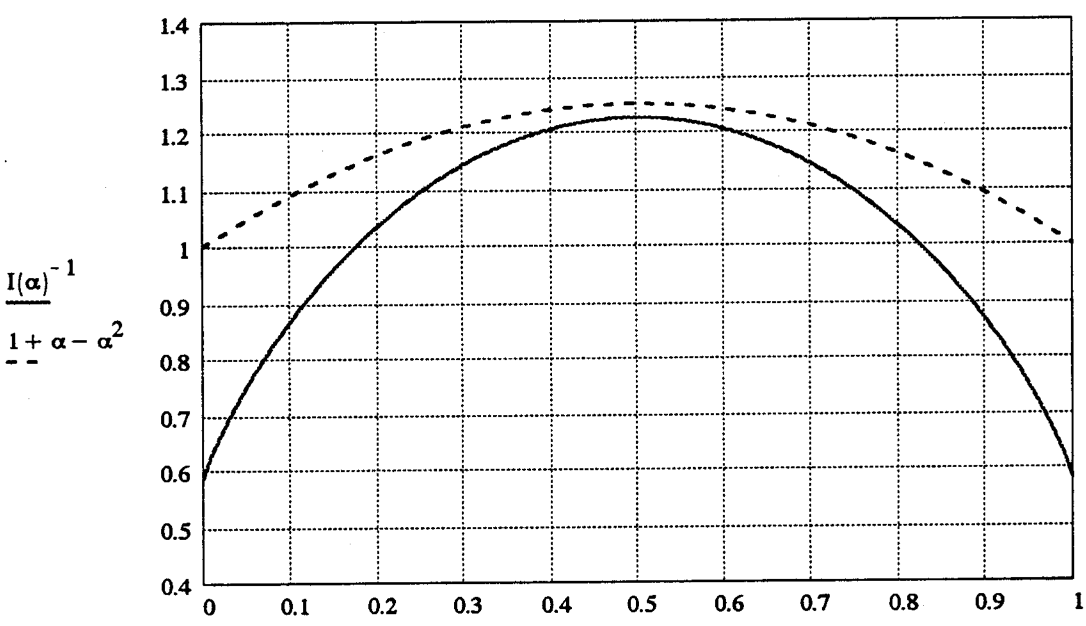
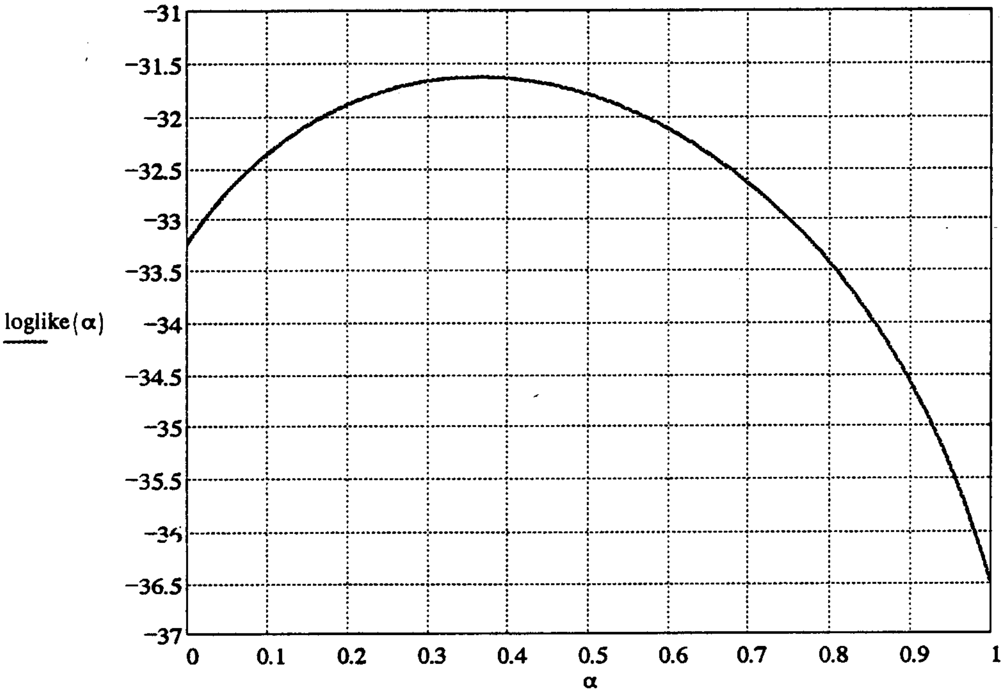


Figure 3



−31
−31.5
−32
−32.5
−33
−33.5
−34
−34.5
−35
−35.5
−36
−36.5
−37
loglike(α)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
α