Stat 543 II

The θ-left-truncated Normal $(\mu, 1)$ distribution has probability density

$$f(x|\theta, \mu) = I[x \geq \theta] \frac{\phi(x-\mu)}{1 - \Phi(\theta - \mu)}$$

for $I[x \geq \theta]$ the indicator that $x \geq \theta$, Φ the standard normal cdf and ϕ the standard normal density. This distribution has mean

$$\Delta(\theta, \mu) = \mu + \frac{\phi(\theta - \mu)}{1 - \Phi(\theta - \mu)} \quad ,$$

and attached to this exam is a plot of $\Delta(\theta, \mu)$ as a function of μ for several different values of θ.

Suppose that $X_1, X_2, ..., X_n$ are iid from this density and inference for $(\theta, \mu) \in \mathcal{R}^2$ is under discussion. Let $M_n = \min(X_1, X_2, ..., X_n)$ and $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and denote realized values of M_n and $\overline{X}_n$ by m_n and $\overline{x}_n$ respectively.

a) Argue carefully that the random vector $(M_n, \overline{X}_n)$ is sufficient for (θ, μ).

Let $L_n(\theta, \mu)$ be the likelihood function and $l_n(\theta, \mu)$ be the loglikelihood here.

b) Argue carefully that for any fixed μ, $\widehat{\theta}_n = m_n$ maximizes $L_n(\theta, \mu)$ as a function of θ.

c) Show that if $\widehat{\mu}_n$ maximizes $L_n(m_n, \mu)$, it must be a solution of the equation

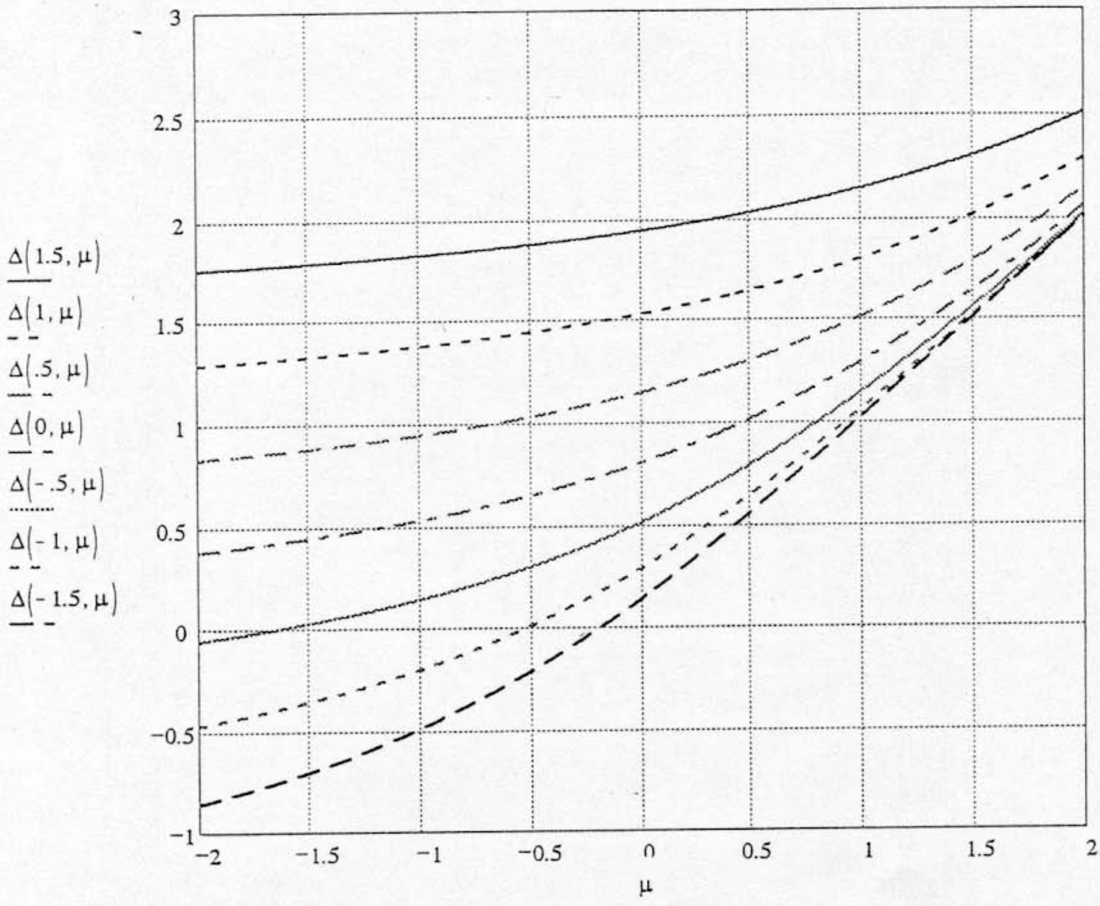
$$\overline{x}_n = \Delta(m_n, \mu) \quad .$$

d) Suppose with $n = 25$, one observes $m_n = .5$ and $\overline{x}_n = 1.5$. Find the MLE of (θ, μ) based on the attached plot of $\Delta(\theta, \mu)$.

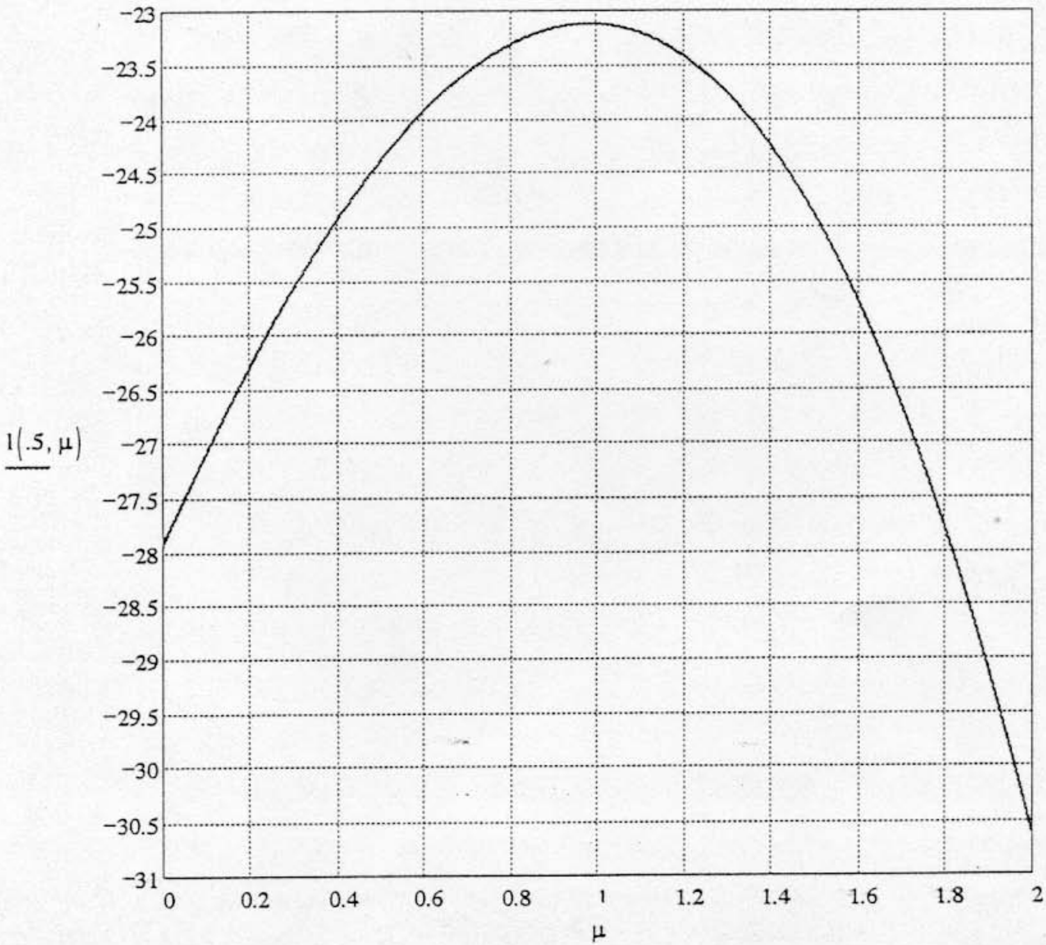
e) Consider testing $\text{H}_0 : (\theta, \mu) = (.25, .8)$ using a likelihood ratio test. Write out an explicit formula for the likelihood ratio statistic for this particular model and null hypothesis.

f) As it turns out, the large n null distribution of $2(l_n(M_n, \widehat{\mu}_n) - l_n(.25, .8))$ is not χ^2_2 as would be expected under standard regularity conditions (which don't hold here). Rather, it is χ^2_3. Attached to this exam is a table of χ^2 quantiles. What can you say about the approximate p-value for the test considered in e) based on the sample referred to in d)?

g) As it turns out, the (θ, μ) large n distribution of $2(l_n(M_n, \widehat{\mu}_n) - l_n(M_n, \mu))$ is χ^2_1 (just as if standard regularity conditions held in this model). Attached to this exam is a plot of $l_n(.5, \mu)$ for the sample referred to in d). Use the plot and this fact to find an approximate 90% confidence interval for μ. (The maximum value of $l_n(.5, \mu)$ shown on the plot is -23.123.)



Δ(1.5, μ)
Δ(1, μ)
Δ(.5, μ)
Δ(0, μ)
Δ(-.5, μ)
Δ(-1, μ)
Δ(-1.5, μ)
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
μ



−23
−23.5
−24
−24.5
−25
−25.5
−26
−26.5
−27
−27.5
−28
−28.5
−29
−29.5
−30
−30.5
−31
l(.5, μ)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
μ