Stat 543 Exam 1

February 18, 1998 Prof. Vardeman

1. Consider a model for n iid discrete observations $X_1, X_2, \ldots, X_n$, each with marginal probability function on $\{0, 1, 2, \ldots\}$

$$f(x|p,\lambda) = \begin{cases} p\exp(-\lambda) + (1-p) & x = 0 \\ p\frac{\exp(-\lambda)\lambda^x}{x!} & x = 1, 2, 3, \ldots \end{cases}$$

where $p \in (0,1)$ and $\lambda > 0$. (This is a mixture of a distribution degenerate at 0 and the Poisson (λ) distribution. It might arise in the inspection for flaws of a mixed lot of items, some of which come from a "perfect" process and others of which come from a process that puts flaws on the items according to a Possion distribution.)

a) Find a two-dimensional sufficient statistic here and argue carefully that it is sufficient.

b) Note that, for example, $\mathrm{E}_{p,\lambda}X_1 = p\lambda$. Using the first two moments, find method of moments estimators for the parameters λ and p.

c) Argue that your estimators for λ and p from b) are consistent.

d) Argue that your estimator of λ from b) is asymptotically normal. (You don't need to work out details.)

e) Give the likelihood equations for this problem and suggest how you might use them in the estimation of λ and p.

2. Consider the estimation of θ based on a single observation X from the (discrete) Uniform distribution on $\{0, 1, 2, \ldots, \theta - 1, \theta\}$. With $\Theta = \{0, 1, 2, \ldots\}$, it is the case that method of moments estimator of θ is $\tilde{\theta}(X) = 2X$, while the maximum likelihood estimator is $\widehat{\theta}(X) = X$.

Two useful facts here are that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

a) Is the statistic X complete? Explain carefully.

b) Is either $\tilde{\theta}(X)$ or $\widehat{\theta}(X)$ an unbiased estimator of θ? Explain.

c) Find both $MSE_\theta(\tilde{\theta}(X))$ and $MSE_\theta(\widehat{\theta}(X))$. (Both are quadratic functions of θ.) Which estimator is preferable in terms of MSE?

3. Below is a statistical model for a discrete observable X, where $\Theta = \{1, 2, 3\}$. (Three different probability mass functions are given, one for each element of Θ.)

	$x = 0$	$x = 1$	$x = 2$	$x = 3$	$x = 4$
$\theta = 1$	.1	.25	.2	.4	.05
$\theta = 2$	.2	.2	.4	.1	.1
$\theta = 3$	.1	.35	.2	.3	.05

Specify any minimal sufficient statistic $T(X)$ here. (Give values of $T(x)$ for $x = 0, 1, 2, 3, 4$.) Argue carefully that the statistic you create is indeed minimal sufficient.

(Hint: Remember that "the shape of the likelihood function is minimal sufficient.")

4. Suppose that $\boldsymbol{X} = (X_1, X_2)$ has (joint) probability density on $[0, 1]^2$

$$f((x_1, x_2)|\theta) = C(\theta)\text{exp}(\theta x_1 + \theta x_2)$$

for an appropriate normalizing constant $C(\theta)$ (don't bother to work this out). For sake of argument, suppose $\Theta = (1, 2)$.

a) The statistic $T(\boldsymbol{X}) = \boldsymbol{X}$ is not complete. Prove this.

b) Identify a complete sufficient statistic here and argue carefully that it has both properties.