Stat 543 Final Exam

May 6, 1998 Prof. Vardeman

1. Consider the simple discrete distribution with probability mass function

$$f(x|p) = \begin{cases} (1-p)p^{x-1} & \text{for } x = 1, 2 \text{ and } 3 \\ p^3 & \text{for } x = 4. \end{cases}$$

This problem concerns hypothesis testing **based on a single observation from this distribution**.

a) Consider the test $\phi(x) = I[x = 1 \text{ or } x = 4]$.

i) Give the power function for this test.

ii) Is this test an "unbiased" test of H$_0$:$p = .5$ vs H$_a$:$p \neq .5$? Explain.

b) Find a most powerful size $\alpha = .15$ test of H$_0$:$p = .1$ vs H$_a$:$p = .2$.

c) For what α's in $[0, 1]$ are there nonrandomized most powerful size α tests of the hypotheses in part b)?

d) There is a UMP size $\alpha = .15$ test of H$_0$:$p \leq .1$ vs H$_a$:$p > .1$. Find it and **argue very carefully that it is indeed UMP size $\boldsymbol{\alpha = .15}$**.

e) Make the Bayesian assumption that *a priori*, p is Uniform $(0, 1)$. The test $\phi(x) = I[x = 3$ or $x = 4]$is a Bayes test for H$_0$:$p \leq .6$ vs H$_a$:$p > .6$. Show that this is correct for $x = 3$ (i.e. that a Bayes test will reject H$_0$ if $x = 3$).

f) What is the size of the test in e)? (**Argue carefully that your "size" is correct**.)

2. Suppose that $X_1, X_2, ..., X_n$ are iid with the marginal distribution specified in question 1.

a) Identify a two-dimensional sufficient statistic here and argue carefully that it is indeed sufficient.

b) Name any unbiased estimator of $p \in [0, 1]$ in this context.

c) Find the maximum likelihood estimator of $p \in [0, 1]$.

3. Consider the model where $X_1, X_2, ..., X_n$ are iid Uniform $[0, \theta]$ for $\theta > 0$. In this model, the largest order statistic $Y_n \doteq \max_{1 \leq i \leq n} X_i$ is sufficient. So if we wish to do hypothesis testing in this model, we need only consider tests based on Y_n.

a) Consider the set of hypotheses H$_0$:$\theta = \theta_0$ vs H$_a$:$\theta > \theta_0$ and tests of the form

$$\phi_k(y) = I[y > k] \quad .$$

Find k_{θ_0} so that $\phi_{k_{\theta_0}}$ is of size $\alpha = .05$.

b) Invert the family of tests $\{\phi_{k_{\theta_0}}\}$ and produce the corresponding 95% confidence procedure for estimating θ.

4. Suppose that for parameters $p_1 \geq 0,\ p_2 \geq 0$ with $p_1 + p_2 \leq 1$, the variables $X_1, X_2, \ldots, X_{40}$ are iid with the simple marginal discrete distribution specified in the following table.

x	1	2	3
$f(x\|p_1,p_2)$	p_1	p_2	$1-p_1-p_2$

However, we are not furnished with the full data set $X_1, X_2, \ldots, X_{40}$, but rather only values of the statistics:

$$N_1 = \sum_{i=1}^{10} I[X_i = 1] \qquad N_{12} = \sum_{i=11}^{20} I[X_i = 1 \text{ or } X_i = 2]$$

$$N_2 = \sum_{i=1}^{10} I[X_i = 2] \qquad N_{13} = \sum_{i=21}^{30} I[X_i = 1 \text{ or } X_i = 3]$$

$$N_3 = \sum_{i=1}^{10} I[X_i = 3] \qquad N_{23} = \sum_{i=31}^{40} I[X_i = 2 \text{ or } X_i = 3]$$

In fact, data in hand are $N_1 = 1,\ N_2 = 4,\ N_3 = 5,\ N_{12} = 4,\ N_{13} = 5$ and $N_{23} = 7$.

a) Write out a loglikelihood function, $\ell(p_1, p_2)$, **based on the *N*'s**.

$\ell(p_1, p_2)$ is complicated enough that pencil-and-paper analysis of it is not sensible. I used MathCad and found out the following about this function of 2 variables. In the first place, it is maximized at $(p_1, p_2) = (.163, .372)$, where $\ell(.163, .372) = -30.652$. Further, for $\ell_{ij}(p_1, p_2)$ the second partial of ℓ with respect to p_i and p_j,

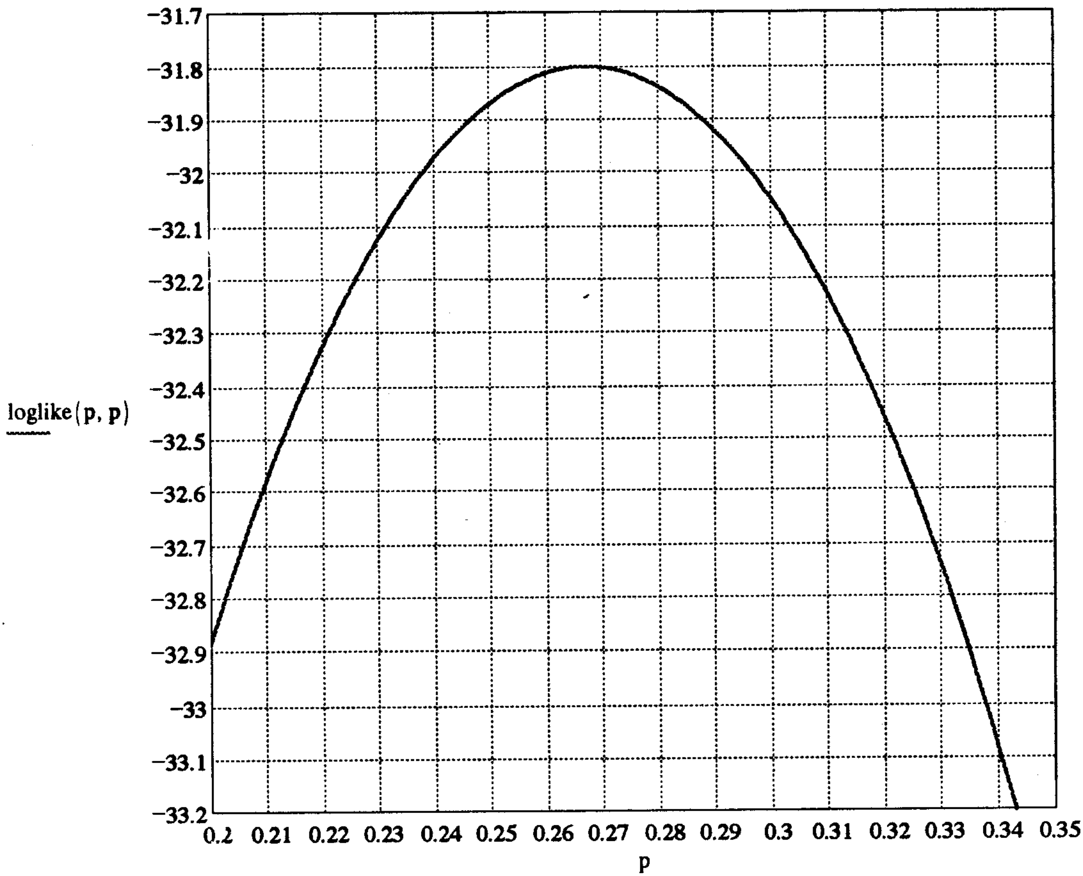
$$\ell_{11}(.163, .372) = -225.7, \quad \ell_{12}(.163, .372) = -64.9, \quad \text{and} \quad \ell_{22}(.163, .372) = -142.5 \ .$$

b) Give a large sample approximate 90% confidence interval for p_1 based on the above information.

Attached to this exam is a plot of $\ell(p, p)$ versus p. (This is the loglikelihood assuming that $p_1 = p_2 = p$.) Use this plot and the information above to do the following.

c) Give a maximum likelihood estimate of p_1 if one supposes that $p_1 = p_2$.

d) Carry out a likelihood ratio test of H_0:$p_1 = p_2$ vs H_a:$p_1 \neq p_2$ at an approximately .05 level. (Since the null hypothesis imposes one restriction on the parameter vector, the appropriate limiting distribution will have 1 degree of freedom.)



−31.7
−31.8
−31.9
−32
−32.1
−32.2
−32.3
−32.4
−32.5
−32.6
−32.7
−32.8
−32.9
−33
−33.1
−33.2
loglike(p, p)
0.2 0.21 0.22 0.23 0.24 0.25 0.26 0.27 0.28 0.29 0.3 0.31 0.32 0.33 0.34 0.35
p