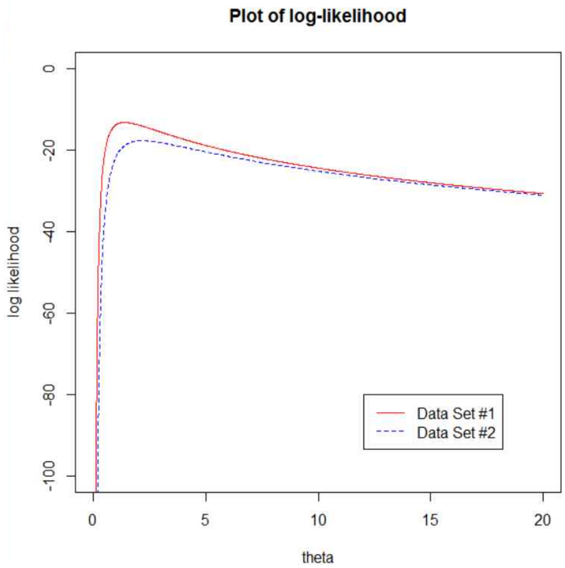
STAT 543

HW 1 Solution

Spring 2016

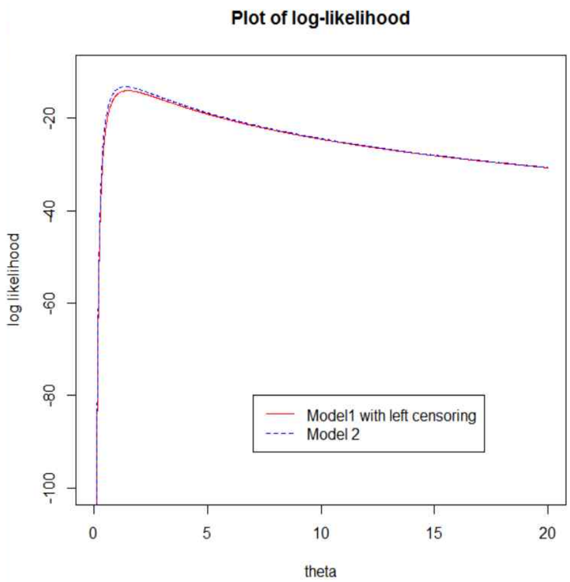
#1.

(a)



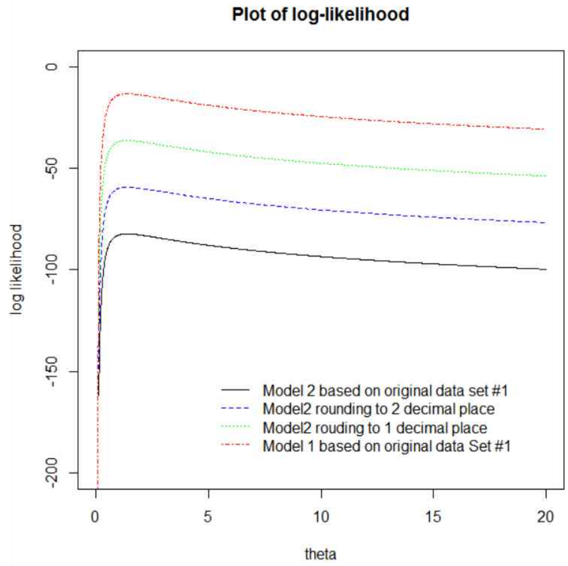
From the plot of the log-likelihood functions based on Data Set #1 and #2, we can see that both curves are almost in the same shape and reach their maximum at around $\theta = 2$, which does provide some reliable indication that both data sets are generated from the given model with $\theta = 2$.

(b)



From the plot above, we can see that the curve of the log-likelihood function of the model with left censoring is in the same shape with the one of the original model; and they almost have the same vertical position.

(c)



Since $F(x|\theta) = 1 - \exp(-x/\theta)$ for $X \sim \mathrm{Exp}(\theta)$,

$$L_r(\theta) = \prod_{i=1}^{10} (F(x_i + r|\theta) - F(x_i - r|\theta))$$

$$= \prod_{i=1}^{10} (\exp(-(x_i - r)/\theta) + \exp(-(x_i + r)/\theta)),$$

where $r = \begin{cases} 0.0005, & \text{for } 3 \ decimal \ place \\ 0.005, & \text{for } 2 \ decimal \ place. \\ 0.05, & \text{for } 1 \ decimal \ place \end{cases}$

The curves of the log-likelihood function of the model II rounding to 3, 2, and 1 decimal place are all in the same pattern with the one of original model I.

3.

① The 1st experiment: $Bin(100, p_{+\,.})$

② The 2nd experiment: $Bin(100, p_{.\,+})$

③ The 3rd experiment: $Multinomial(50, p_{++}, p_{+\,-}, p_{-\,+}),\ p_{--} = 1 - (p_{++} + p_{+\,-} + p_{-\,+})$

where $p_{+\,.} = p_{++} + p_{+\,-}$ and $p_{.\,+} = p_{++} + p_{-\,+}$.

Since these three experiments are mutually independent,

$$\log L(p|x) = \log\left(\frac{100!}{10!90!}(p_{+\,.})^{10}(1-p_{+\,.})^{90}\right) + \log\left(\frac{100!}{20!80!}(p_{.\,+})^{10}(1-p_{.\,+})^{90}\right)$$
$$+\log\left(\frac{50!}{4!2!8!36!}(p_{++})^4(p_{+\,-})^2(p_{-\,+})^8(1-p_{++}-p_{+\,-}-p_{-\,+})^{36}\right).$$

4.

① B&D 1.2.2.

(a)

$$\pi(\theta|x) = \frac{p(x|\theta)\pi(\theta)}{\int_\Theta p(x|\theta)\pi(\theta)d\theta} = \frac{\frac{2x}{\theta^2}I(0<x<\theta)I(0\le\theta\le 1)}{\int_x^1 \frac{2x}{\theta^2}d\theta} = \frac{\frac{2x}{\theta^2}I(0<x<\theta\le 1)}{\left[\frac{-2x}{\theta}\right]_x^1}$$

$$= \frac{x}{\theta^2(1-x)}I(0<x<\theta\le 1)$$

(b)

$$\pi(\theta|x) = \frac{p(x|\theta)\pi(\theta)}{\int_\Theta p(x|\theta)\pi(\theta)d\theta} = \frac{\frac{2x}{\theta^2}I(0<x<\theta)(3\theta^2)I(0\le\theta\le 1)}{\int_x^1 \frac{2x}{\theta^2}3\theta^2 d\theta} = \frac{6x\,I(0<x<\theta\le 1)}{[2x(3)\theta]_x^1}$$

$$= \frac{6x}{6x(1-x)}I(0<x<\theta\le 1) = \frac{1}{(1-x)}I(0<x<\theta\le 1)$$

(c)

For the prior in (a),

$$E(\theta|X) = \int_x^1 \theta\frac{2x}{\theta^2(2-2x)}\,d\theta = \int_x^1 \frac{2x}{(2-2x)}\,\frac{1}{\theta}d = \frac{2x}{(2-2x)}\,[\log\theta]_x^1$$

$$= \frac{x}{(1-x)}(-\log x),\ \ 0<x<1.$$

For the prior in (b),

$$E(\theta|X) = \int_x^1 \theta\frac{1}{(1-x)}\,d\theta = \frac{1}{(1-x)}\left[\frac{1}{2}\theta^2\right]_x^1 = \frac{1-x^2}{2(1-x)}$$

$$= \frac{1+x}{2},\ \ 0<x<1.$$

(d)

$$\pi(\theta|\boldsymbol{x}) = \frac{p(\boldsymbol{x}|\theta)\pi(\theta)}{\int_{\Theta} p(\boldsymbol{x}|\theta)\pi(\theta)d\theta} = \frac{\prod_{i=1}^{n}\left(\frac{2x_i}{\theta^2}\right)I(0<x_i<\theta)I(0<\theta<1)}{\int_{\Theta}\prod_{i=1}^{n}\left(\frac{2x_i}{\theta^2}\right)I(0<x_i<\theta)d\theta}$$

$$= \frac{\left(\frac{\prod_{i=1}^{n}2x_i}{\theta^{2n}}\right)I(x_{(n)}<\theta<1)}{\int_{x_{(n)}}^{1}\left(\frac{\prod_{i=1}^{n}2x_i}{\theta^{2n}}\right)d\theta} = \frac{\left(\frac{\prod_{i=1}^{n}2x_i}{\theta^{2n}}\right)I(x_{(n)}<\theta<1)}{\prod_{i=1}^{n}2x_i\left[\frac{1}{1-2n}\theta^{1-2n}\right]_{x_{(n)}}^{1}}$$

$$= \frac{\left(\frac{\prod_{i=1}^{n}x_i}{\theta^{2n}}\right)I(x_{(n)}<\theta<1)}{\prod_{i=1}^{n}x_i\frac{1}{1-2n}(1-x_{(n)}^{1-2n})} = \frac{(1-2n)}{\theta^{2n}(1-x_{(n)}^{1-2n})}I(x_{(n)}<\theta<1),$$

where $x_{(n)}$ is the maximum of x_1, ..., x_n.

② B&D 1.2.3.

(a) Given the prior distribution of θ, $\pi(\theta)=\frac{1}{3}$ for $\theta=\frac{1}{4},\frac{1}{2},\frac{3}{4}$.

The posterior density of θ given $X=2$ is as follows.

$$\pi(\theta|x=2) = \frac{p(x=2|\theta)\pi(\theta)}{\sum_{\Theta}p(x=2|\theta)\pi(\theta)}$$

$$= \frac{(1-\theta)^2\theta(1/3)}{\left((1-1/4)^2(1/4)+(1-1/2)^2(1/2)+(1-3/4)^2(3/4)\right)(1/3)} = \frac{(1-\theta)^2\theta}{5/16},$$

where $\theta=\frac{1}{4},\frac{1}{2}$, or $\frac{3}{4}$.

(b)

$$\pi(\theta|x=2) = \begin{cases} 9/20\ ,\theta=1/4 \\ 2/5\ \ ,\theta=1/2 \\ 3/20\ ,\theta=3/4 \end{cases}$$

The most probable value of θ given $x=2$ is $\theta=1/4$.

Given $x=k$,

$$\pi(\theta|x=2)=\frac{p(x=k|\theta)\pi(\theta)}{\sum_{\Theta}p(x=k|\theta)\pi(\theta)}$$

$$=\frac{(1-\theta)^k\theta(1/3)}{\left((1-1/4)^k(1/4)+(1-1/2)^k(1/2)+(1-3/4)^k(3/4)\right)(1/3)}$$

$$=\frac{4^{k+1}(1-\theta)^k\theta}{3^k+2^{k+1}+3}$$

$$=\begin{cases}(3^k/(3^k+2^{2k+1}+3) \ , & \theta=1/4\\ (2^{k+1}/(3^k+2^{2k+1}+3), & \theta=1/2.\\ (3/(3^k+2^{2k+1}+3) \ , & \theta=3/4\end{cases}$$

The most probable values of θ given $x=k$ are $\theta=3/4$ if $k=0$, $\theta=1/2$ if $k=1$ and $\theta=1/4$ if $k\geq 2$.

(c)

$$\pi(\theta|X=k)=\frac{p(X=k|\theta)\pi(\theta)}{\int_{\Theta}p(X=k|\theta)\pi(\theta)\,d\theta}=\frac{(1-\theta)^k\theta\frac{1}{\beta(r,s)}\theta^{r-1}(1-\theta)^{s-1}I(0<\theta<1)}{\int_0^1(1-\theta)^k\theta\frac{1}{\beta(r,s)}\theta^{r-1}(1-\theta)^{s-1}d\theta}$$

$$=\frac{(1-\theta)^{s+k-1}\theta^r I(0<\theta<1)}{\int_0^1(1-\theta)^{s+k-1}\theta^r\,d\theta}=\frac{1}{\beta(r+1,s+k)}(1-\theta)^{s+k-1}\theta^r\ I(0<\theta<1)$$

Therefore, $\theta|X=k\sim Beta(r+1,s+k)$.

③ B&D 1.2.14.

(a)

$$f(x_{n+1}|\theta)=\frac{1}{\sqrt{2\pi}\,\sigma_0}exp\left\{-\frac{(x_{n+1}-\theta)^2}{2\sigma_0^2}\right\}$$ where $$\pi(\theta)\frac{1}{\sqrt{2\pi}\,\tau_0}exp\left\{-\frac{(\theta-\theta_0)^2}{2\tau_0^2}\right\}$$

$$f(x_{n+1})=\int_{-\infty}^{\infty}f(x_{n+1}|\theta)\pi(\theta)d\theta$$

$$=\frac{1}{2\pi\sigma_0\tau_0}\int_{-\infty}^{\infty}\exp\left\{-\frac{(x_{n+1}-\theta)^2}{2\sigma_0^2}-\frac{(\theta-\theta_0)^2}{2\tau_0^2}\right\}d\theta$$

$$=\frac{1}{2\pi\sigma_0\tau_0}\int_{-\infty}^{\infty}\exp\left\{-\frac{\left(\theta-\frac{\tau_0^2x_{n+1}+\sigma_0^2\theta_0}{\sigma_0^2+\tau_0^2}\right)^2+\frac{\sigma_0^2\tau_0^2(x_{n+1}-\theta_0)^2}{(\sigma_0^2+\tau_0^2)^2}}{2\frac{\sigma_0^2\tau_0^2}{\sigma_0^2+\tau_0^2}}\right\}d\theta$$

$$= \frac{1}{\sqrt{2\pi}\ \sqrt{\sigma_0^2+\tau_0^2}} exp\left\{-\frac{(x_{n+1}-\theta_0)^2}{2\,(\sigma_0^2+\tau_0^2)}\right\}.$$

Therefore, $X_{n+1} \sim N(\theta_0, \tau_0^2+\sigma_0^2)$.

Since

$$f(x_1,\ \cdots, x_{n+1}|\theta) = \prod_{i=1}^{n+1} \frac{1}{\sqrt{2\pi}\,\sigma_0} exp\left[-\frac{(x_i-\theta)^2}{2\sigma_0^2}\right],$$

and

$$\pi(\theta) = \frac{1}{\sqrt{2\pi}\,\tau_0} exp\left[-\frac{(\theta-\theta_0)^2}{2\tau_0^2}\right],$$

$$f(x_1,\ \cdots, x_{n+1}) = \int_{-\infty}^{\infty} f(x_1,\ \cdots, x_{n+1}|\theta)\pi(\theta)\,d(\theta)$$

$$= \frac{1}{(2\pi\sigma_0^2)^{(n+1)/2}\tau_0} \int_{-\infty}^{\infty} \exp\left[-\frac{\sum_{i=1}^{n+1}(x_i-\theta)^2}{2\sigma_0^2}\right] \exp\left[-\frac{(\theta-\theta_0)^2}{2\tau_0^2}\right] d(\theta)$$

$$= \frac{\int_{-\infty}^{\infty} \exp\left[-\frac{\left(\theta-\frac{\tau_0^2\sum_{i=1}^{n+1}x_i+\sigma_0^2\theta_0}{(n+1)\tau_0^2+\sigma_0^2}\right)^2+\frac{(n+1)\tau_0^2\sum_{i=1}^{n+1}(x_i-\overline{x})^2+\sigma_0^2\tau_0\sum_{i=1}^{n+1}(x_i-\theta_0)^2}{(n+1)\tau_0^2+\sigma_0^2}}{2\frac{\tau_0^2\sigma_0^2}{(n+1)\tau_0^2+\sigma_0^2}}\right] d(\theta)}{(2\pi\sigma_0^2)^{(n+1)/2}\tau_0}$$

$$= \frac{1}{(2\pi)^{(n)/2}\ \sigma_0^n \sqrt{(n+1)\tau_0^2+\sigma_0^2}} exp\left[-\frac{(n+1)\,\tau_0^4\sum_{i=1}^{n+1}(x_i-\overline{x_{n+1}})^2+\sigma_0^2\tau_0^2\sum_{i=1}^{n+1}(x_i-\theta_0)^2}{2\,((n+1)\tau_0^2+\sigma_0^2)^2}\right].$$

Similarly,

$$f(x_1,\ \cdots, x_n) = \frac{1}{(2\pi)^{(n-1)/2}\ \sigma_0^{n-1}\sqrt{n\tau_0^2+\sigma_0^2}} exp\left[-\frac{n\tau_0^4\sum_{i=1}^{n}(x_i-\overline{x_n})^2+\sigma_0^2\tau_0^2\sum_{i=1}^{n}(x_i-\theta_0)^2}{2\,((n)\tau_0^2+\sigma_0^2)^2}\right].$$

Thus, $f(x_{n+1}|x_1,\ \cdots, x_n) = \frac{f(x_1,\ \cdots, x_{n+1})}{f(x_1,\ \cdots, x_n)} = \frac{\sqrt{n\sigma_0^2+\tau_0^2}}{\sqrt{2\pi}\ \sigma_0\sqrt{(n+1)\sigma_0^2+\tau_0^2}}$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\sigma_0^2\left(1+\frac{\tau_0^2}{n\tau_0^2+\sigma_0^2}\right)}} exp\left[-\frac{\left(x_{n+1}-\frac{n\tau_0^2\overline{x_n}+\sigma_0^2\theta_0}{n\tau_0^2+\sigma_0^2}\right)^2}{2\sigma_0^2\left(1+\frac{\tau_0^2}{n\tau_0^2+\sigma_0^2}\right)}\right].$$

Then, the posterior predictive distribution is as follows.

$$X_{n+1}|X_1, \cdots, X_n \sim N\left(\frac{n\tau_0^2\overline{x_n}+\sigma_0^2\theta_0}{n\tau_0^2+\sigma_0^2}, \sigma_0^2\left(1+\frac{\tau_0^2}{n\tau_0^2+\sigma_0^2}\right)\right)$$

Solution 2

Consider $X_i = \theta + \epsilon_i$, where $\epsilon_i \sim iid\ N(0,\sigma_0^2)$, and $\theta \sim N(\theta_0, \tau_o^2)$ and ϵ_i and θ are independent. Then,

$$(X_1, X_2, \cdots, X_n, X_{n+1}, \theta)^T \sim MVN(\theta_0 \mathbf{1}, \Sigma),$$

where $\Sigma = \sigma_0^2 I_{(n+2)\times(n+2)} + \tau_0^2 J_{(n+2)\times(n+2)}$ and J is a matrix of 1's.

So,

$$(X_1, X_2, \cdots, X_n)^T \sim MVN(\theta_0 \mathbf{1}, \Sigma_{XX}),$$

where $\Sigma_{XX} = \sigma_0^2 I_{n\times n} + \tau_0^2 J_{n\times n}$. Also, $X_{n+1} \sim N(\theta_0, \sigma_0^2 + \tau_0^2)$.

Let $W = (X_1, \cdots, X_n)^T$ and $Z = X_{n+1}$. Then $W \sim MVN(\theta_0 \mathbf{1}, \Sigma_{XX})$, $Z \sim N(\theta_0, \sigma_0^2 + \tau_0^2)$ and $\Sigma_{WZ} = \tau_0^2 \mathbf{1}_{(n-1)\times 1}$.

By using the formula of

$$E(Z\,|\,W) = \mu_Z + \Sigma_{WZ}\Sigma_{WW}^{-1}(W - \mu_W),$$

and

$$Cov(Z\,|\,W) = \Sigma_{ZZ} - \Sigma_{ZW}\Sigma_{WW}^{-1}\Sigma_{WZ}\ ,$$

we can get

$$E(X_{n+1}|X_1, \cdots, X_n) = \frac{n\tau_0^2\overline{x}_n + \sigma_0^2\theta_0}{n\tau_0^2 + \sigma_0^2}$$

and

$$Var(X_{n+1}|X_1, \cdots, X_n) = \sigma_0^2(1 + \frac{\tau_0^2}{n\tau_0^2 + \sigma_0^2}).$$

Hence, $X_{n+1}|X_1, \cdots, X_n \sim N(\frac{n\tau_0^2\overline{x}_n + \sigma_0^2\theta_0}{n\tau_0^2 + \sigma_0^2}, \sigma_0^2(1 + \frac{\tau_0^2}{n\tau_0^2 + \sigma_0^2})).$

* Notice that the inverse of a matrix in the form of

$$aI + bJ$$

has the form of

$$\alpha I+\beta J$$

for $\alpha=\dfrac{1}{a}$ and $\beta=-\dfrac{b}{a(a+kb)}$, where k is the dimension of the square matrix I (or J).

(b)

As $n\rightarrow\infty$, the posterior predictive distribution becomes close to $N(\theta_0,\sigma_0^2)$, while the predictive distribution of X_{n+1} is still $N(\theta_0,\tau_0^2+\sigma_0^2)$.

④ B&D 1.3.3.

(a)

$R(\theta,\delta_{r,s})=E_\theta(L(\theta,\delta_{r,s}(X)))$

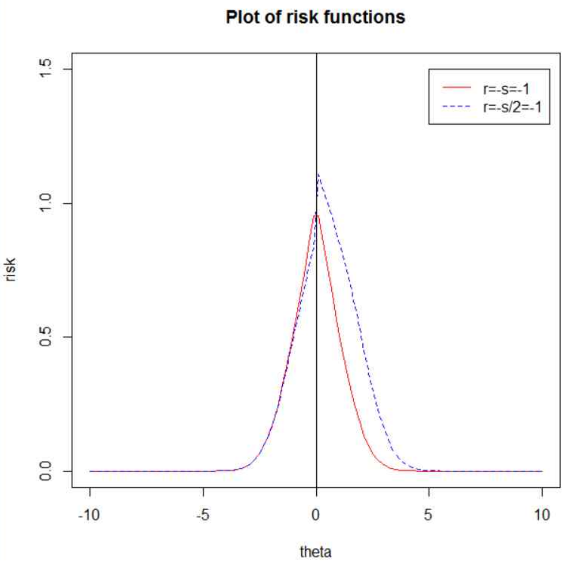
$$=\begin{cases}0\,P(\overline{X}< r)+cP(r<\overline{X}< s)+(b+c)P(\overline{X}> s) & ,\theta<0\\ bP(\overline{X}< r)+0P(r<\overline{X}< s)+bP(\overline{X}> s) & ,\theta=0\\ (b+c)\,P(\overline{X}< r)+cP(r<\overline{X}< s)+0(b+c)P(\overline{X}> s) & ,\theta>0\end{cases}$$

$$=\begin{cases}c(\Phi(\sqrt{n}\,(s-\theta))-\Phi(\sqrt{n}\,(r-\theta)))+(b+c)(1-\Phi(\sqrt{n}\,(s-\theta))) & ,\theta<0\\ b\Phi(\sqrt{n}\,(r-\theta))+b(1-\Phi(\sqrt{n}\,(s-\theta))) & ,\theta=0\\ (b+c)\Phi(\sqrt{n}\,(r-\theta))+c(\Phi(\sqrt{n}\,(s-\theta))-\Phi(\sqrt{n}\,(r-\theta))) & ,\theta>0\end{cases}$$

$$=\begin{cases}c\overline{\Phi}(\sqrt{n}\,(r-\theta))+b\overline{\Phi}(\sqrt{n}\,(s-\theta)) & ,\theta<0\\ b\overline{\Phi}(\sqrt{n}\,s)+b\Phi(\sqrt{n}\,r) & ,\theta=0\\ b\Phi(\sqrt{n}\,(r-\theta))+c\Phi(\sqrt{n}\,(s-\theta)) & ,\theta>0\end{cases}$$

where $\overline{\Phi}=1-\Phi$ and Φ is the normal cdf.

(b)



From the plot, we can see that when $\theta > 0$, the procedure with $r = -s = -1$ has smaller risk than the procedure with $r = -s/2 = -1$.

⑤ B&D 1.3.9.

$X \sim Bin(n, \theta_0)$,

$$E(\hat{p}) = E\left(\frac{X}{n}\right) = \theta_0 \ , \ \ Var(\hat{p}) = Var\left(\frac{X}{n}\right) = \frac{\theta_0(1-\theta_0)}{n},$$

$E(\hat{\theta}) = (.2)(.10) + (.8)E(\hat{p}) = 0.02 + 0.8\theta_0$ and $Var(\hat{\theta}) = (0.8)^2 Var(\hat{p}) = 0.64\frac{\theta_0(1-\theta_0)}{n}$

$$MSE(\hat{p}) = \frac{\theta_0(1-\theta_0)}{n}, \ MSE(\hat{\theta}) = (0.2\theta_0 - 0.02)^2 + 0.64\frac{\theta_0(1-\theta_0)}{b=n}$$

$$\frac{MSE(\hat{\theta})}{MSE(\hat{p})} < 1 \ \Rightarrow (0.2\theta_0 - 0.02)^2 + 0.64\frac{\theta_0(1-\theta_0)}{n} < \frac{\theta_0(1-\theta_0)}{n}$$

$$\Rightarrow (n+9)\theta_0^2 - (0.2n+9)\theta_0 + 0.01n < 0$$

$0.0187 < \theta_0 < 0.3931$ for $n = 25$,

$0.0407 < \theta_0 < 0.2253$ for $n = 100$.

⑥ B&D 1.3.10.

From part (a) in problem 1.3.3,

$$E_\theta(L(\theta, \delta_{r,s}(x)) = \begin{cases} c\overline{\Phi}(\sqrt{n}(r-\theta)) + b\overline{\Phi}(\sqrt{n}(s-\theta)) & , \theta < 0 \\ b\overline{\Phi}(\sqrt{n}s) + b\Phi(\sqrt{n}r) & , \theta = 0 \\ b\Phi(\sqrt{n}(r-\theta)) + c\Phi(\sqrt{n}(s-\theta)) & , \theta > 0 \end{cases}$$

$$= \begin{cases} \overline{\Phi}((r-\theta)) + \overline{\Phi}((s-\theta)) & , \theta < 0 \\ \overline{\Phi}(s) + \Phi(r) & , \theta = 0 \\ \Phi((r-\theta)) + \Phi((s-\theta)) & , \theta > 0 \end{cases}$$

where $\overline{\Phi} = 1 - \Phi$ and Φ is the standard normal distribution function and the prior distribution is as follows.

$$\pi(0) = \pi(-1/2) = \pi(1/2) = 1/3.$$

(a)

$$E_\theta(L(\theta, \delta_{r=-1, s=1}(x)) = \begin{cases} \overline{\Phi}((-1-\theta)) + \overline{\Phi}((1-\theta)) & , \theta < 0 \\ \overline{\Phi}(1) + \Phi(-1) & , \theta = 0 \\ \Phi((-1-\theta)) + \Phi((1-\theta)) & , \theta > 0 \end{cases}$$

The the Bayes risk of $\delta_{-1,1}(x)$ is

$R(G,\delta_{-1,1}(x))=\sum_{\Theta}E_{\theta}(L(\theta,\delta_{-1,1}(x))\pi(\theta)$

$$=\frac{\left\{\overline{\Phi}(-\frac{1}{2})+\overline{\Phi}(\frac{3}{2})+(\overline{\Phi}(1)+\Phi(-1))+\Phi(\frac{-3}{2})+\Phi(\frac{1}{2})\right\}}{3}$$

$=0.61128$

(b)

$$E_{\theta}(L(\theta,\delta_{r=-1,s=2}(x))=\begin{cases}\overline{\Phi}((-1-\theta))+\overline{\Phi}((2-\theta)) & ,\theta<0\\ \overline{\Phi}(2)+\Phi(-1) & ,\theta=0\\ \Phi((-1-\theta))+\Phi((2-\theta)) & ,\theta>0\end{cases}$$

The the Bayes risk of $\delta_{-1,2}(x)$ is

$R(G,\delta_{-1,2}(x))=\sum_{\Theta}E_{\theta}(L(\theta,\delta_{-1,2}(x))\pi(\theta)$

$$=\frac{\left\{\overline{\Phi}(-\frac{1}{2})+\overline{\Phi}(\frac{5}{2})+(\overline{\Phi}(2)+\Phi(-1))+\Phi(\frac{-3}{2})+\Phi(\frac{3}{2})\right\}}{3}$$

$=0.626359.$

From the Bayes point of view, $\delta_{-1,1}(x)$ is a better decision rule.

⑨ B&D 1.3.18.

(a)

Risk function is $R(\theta,\delta_k)=L(\theta,1)P(\delta=1)+L(\theta,0)P(\delta=0)$

$$=\begin{cases}sP_{\theta}(X\geq k)+rN\theta P_{\theta}(X<k) & ,\theta<\theta_0\\ rN\theta P_{\theta}(X<k) & \theta>\theta_0\end{cases}$$

where the loss function is $\begin{cases}L(\theta,1)=s & ,\theta<\theta_0\\ L(\theta,1)=0 & ,\theta\geq\theta_0.\\ l(\theta,0)=rN\theta\end{cases}$

(b)

For $N=10$, $s=r=1$, $\theta_0=.1$ and $k=3$

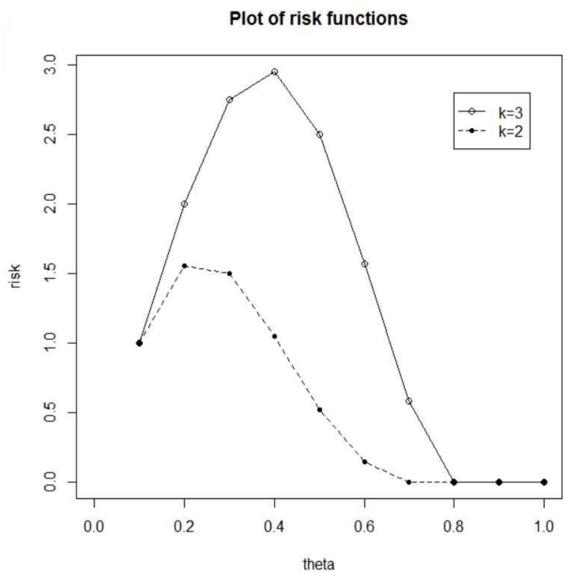
$$R(\theta,\delta_3)=\begin{cases}P_{\theta}(X\geq 3)+10\theta P_{\theta}(X<3) & ,\theta<0.1\\ 10\theta P_{\theta}(X<3) & \theta\geq 0.1\end{cases}.$$

(c)

For $N=10,\ s=r=1,\ \theta_0=.1$ and $k=2$

$$R(\theta,\delta_2)=\begin{cases}P_\theta(X\geq 2)+10\theta P_\theta(X<2), & \theta<0.1\\ 10\theta P_\theta(X<2) & \theta\geq 0.1\end{cases}$$

, where $P(X<k)=\sum_{i=0}^{k-1}\frac{\binom{N\theta}{i}\binom{N-N\theta}{n-i}}{\binom{N}{n}}$ and $\max\{\mathrm{n}-\mathrm{N}(1-\theta),0\}\leq \mathrm{i}\leq \min\{\mathrm{N}\theta,\mathrm{n}\}$



From the plot, $R(\theta,\delta_2)$ is smaller than $R(\theta,\delta_3)$ over all possible values of θ, which means that δ_2 is a better decision rule.