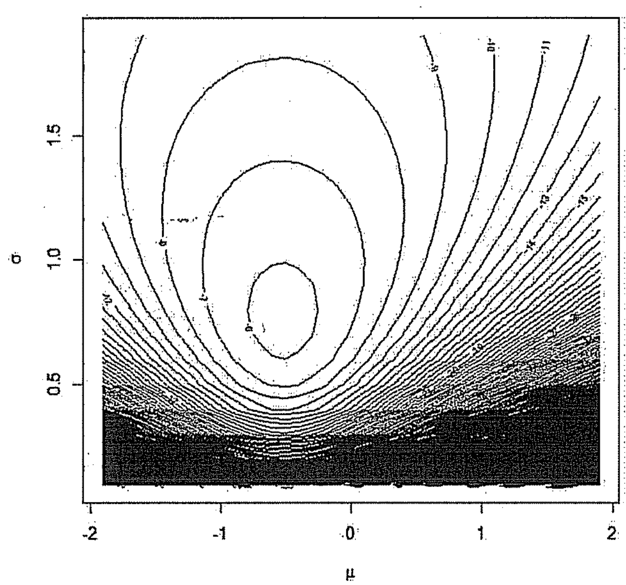
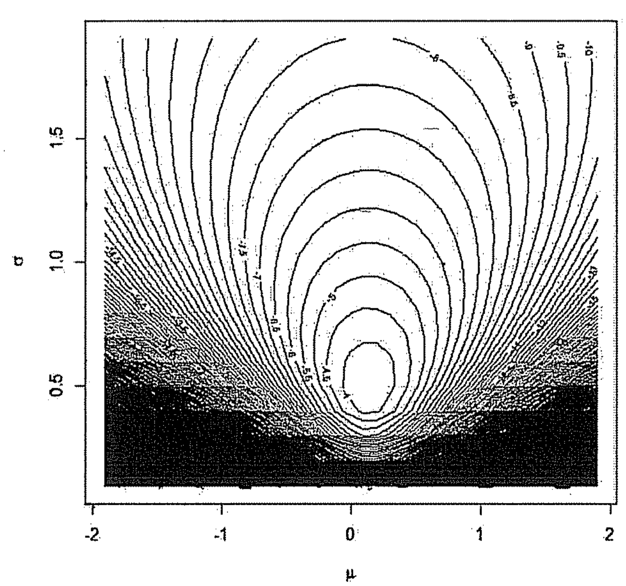
STAT 543
HW 2 Solution
Spring 2016

#1.

Data Set1



Data Set2



#1.3.11

(a)

Pf.) Since $E_\theta[l(\theta, \delta(\underset{\sim}{X}))] = E_\theta[(\theta - \delta(\underset{\sim}{X}))^2] = Var_\theta\, \delta(\underset{\sim}{X}) + [E_\theta \delta(\underset{\sim}{X}) - \theta]^2$,

it is enough to show that

$$[E_\theta \delta(\underset{\sim}{X}) - \theta]^2 \le [E_\theta \delta(\underset{\sim}{X}) - \theta']^2, \quad \forall \theta, \theta' \in \Theta \iff E_\theta \delta(\underset{\sim}{X}) = \theta.$$

$\Leftarrow$: $0 \le [E_\theta \delta(\underset{\sim}{X}) - \theta']^2$ holds.

$\Rightarrow$: Suppose that $[E_\theta \delta(\underset{\sim}{X}) - \theta]^2 \le [E_\theta \delta(\underset{\sim}{X}) - \theta']^2$ for $\forall \theta, \theta' \in \Theta$

For $\theta \ne \theta'$,

$$[E_\theta \delta(\underset{\sim}{X}) - \theta]^2 \le [E_\theta \delta(\underset{\sim}{X}) - \theta']^2 = [E_\theta \delta(\underset{\sim}{X}) - \theta + \theta - \theta']^2$$

$$= [E_\theta \delta(\underset{\sim}{X}) - \theta]^2 + 2(E_\theta \delta(\underset{\sim}{X}) - \theta)(\theta - \theta') + (\theta - \theta')^2, \quad \forall \theta, \theta' \in \Theta$$

$$\Rightarrow \quad 0 \le 2(E_\theta \delta(\underset{\sim}{X}) - \theta)(\theta - \theta') + (\theta - \theta')^2, \quad \forall \theta, \theta' \in \Theta \text{ with } \theta \ne \theta'$$

(This is a quadratic inequality w.r.t. $\theta - \theta'$)

In order to hold the inequality for any $\theta - \theta'$,

$(E_\theta \delta(\underset{\sim}{X}) - \theta)^2 \le 0$ i.e., $E_\theta \delta(\underset{\sim}{X}) = \theta$.

Or,

(Easy way : Take $\theta' = E_\theta \delta(\underset{\sim}{X})$.

Since $[E_\theta \delta(\underset{\sim}{X}) - \theta]^2 \le [E_\theta \delta(\underset{\sim}{X}) - \theta']^2 = 0$, $E_\theta \delta(\underset{\sim}{X}) = \theta$) //

(b)

p.f) claim: $E_\theta \, l(\theta, \delta(\underline{x})) \le E_\theta \, l(\theta', \delta(\underline{x}))$, $\forall \theta, \theta' \in \Theta$ $\iff$ $E_{\theta'} \delta(\underline{x}) \ge \sup_{\theta \in \Theta_0} E_\theta \delta(\underline{x})$

$\Leftarrow$: Not true, in general. with the 0-1 loss fct.

$\Rightarrow$: Suppose that $E_\theta \, l(\theta, \delta(\underline{x})) \le E_\theta \, l(\theta', \delta(\underline{x}))$, $\forall \theta, \theta' \in \Theta$.

Then, for any $\theta \in \Theta_0$ and $\theta' \in \Theta_1$,

$$P_\theta (\delta(\underline{x}) = 1) \le P_\theta(\delta(\underline{x}) = 0) = 1 - P_\theta(\delta(\underline{x}) = 1) \qquad \therefore \; P_\theta(\delta(\underline{x}) = 1) \le \frac{1}{2}$$

Pick any $\theta'' \in \Theta_0$:

Since $E_{\theta'} \, l(\theta', \delta(\underline{x})) \le E_{\theta'} \, l(\theta'', \delta(\underline{x}))$,

$$P_{\theta'}(\delta(\underline{x}) = 0) \le P_{\theta'}(\delta(\underline{x}) = 1)$$

$$\iff \quad 1 - P_{\theta'}(\delta(\underline{x}) = 1) \le P_{\theta'}(\delta(\underline{x}) = 1) \qquad \therefore \; P_{\theta'}(\delta(\underline{x}) = 1) \ge \frac{1}{2}$$

$\therefore$ For any $\theta' \in \Theta_1$,

$$\beta(\theta', \delta) = E_{\theta'} \delta(\underline{x}) = P_{\theta'}(\delta(\underline{x}) = 1) \ge \frac{1}{2} \ge P_\theta(\delta(\underline{x}) = 1), \quad \forall \theta \in \Theta_0.$$

i.e., $\beta(\theta', \delta) \ge \sup_{\theta \in \Theta_0} \beta(\theta, \delta)$, $\forall \theta' \in \Theta_1$ //

(*) If we use some particular loss fcts, we may be able to prove "$\Leftarrow$".

Ex). Consider $l(\theta, a) = \begin{cases} 0 & \\ \alpha & \theta \in \Theta_1, \; a = 0 \\ 1-\alpha & \theta \in \Theta_0, \; a = 1 \end{cases}$, $\alpha = \sup_{\theta \in \Theta_0} \beta(\theta, \delta) > 0$,

Then, $E_\theta \, l(\theta', \delta(\underline{x})) = \begin{cases} (1-\alpha) P_\theta(\delta(\underline{x}) = 1), & \theta' \in \Theta_0 \\ \alpha \, P_\theta(\delta(\underline{x}) = 0), & \theta' \in \Theta_1 \end{cases} = \begin{cases} (1-\alpha) \beta(\theta, \delta), & \theta' \in \Theta_0 \\ \alpha(1 - \beta(\theta, \delta)); & \theta' \in \Theta_1 \end{cases}$

Now, suppose $\beta(\theta', \delta) \ge \sup_{\theta \in \Theta_0} \beta(\theta, \delta) = \alpha$, $\forall \theta' \in \Theta_1$. where $\alpha > 0$.

Then, $\beta(\theta, \delta) \geq \alpha, \quad \forall \theta \in \Theta_1 \quad \Rightarrow \quad 1-\beta(\theta, \delta) \leq 1-\alpha, \quad \forall \theta \in \Theta_1 \quad \cdots ①$

$\beta(\theta, \delta) \leq \alpha, \quad \forall \theta \in \Theta_0 \quad \Rightarrow \quad 1-\beta(\theta, \delta) \geq 1-\alpha, \quad \forall \theta \in \Theta_0 \quad \cdots ②$

$$\Rightarrow \quad E_\theta\, \ell(\theta, \delta(\underset{\sim}{x})) = (1-\alpha)\beta(\theta, \delta) \overset{②}{\leq} (1-\alpha)\cdot\alpha \leq \alpha\cdot(1-\beta(\theta, \delta)) = E_\theta\, \ell(\theta', \delta(\underset{\sim}{x})), \quad \forall \theta \in \Theta_0, \theta' \in \Theta_1$$

$$E_\theta\, (\ell(\theta', \delta(\underset{\sim}{x})) = (1-\alpha)\beta(\theta, \delta) \overset{①}{\geq} (1-\alpha)\cdot\alpha \geq \alpha\, (1-\beta(\theta, \delta)) = E_\theta\, \ell(\theta, \delta(\underset{\sim}{x})), \quad \theta \in \Theta_1, \theta' \in \Theta_0$$

If both of θ and θ' are on the same subspace (either Θ_0 or Θ_1),

$$E_\theta\, \ell(\theta, \delta(\underset{\sim}{x})) = E_\theta\, \ell(\theta', \delta(\underset{\sim}{x}))$$

Thus, $E_\theta\, \ell(\theta, \delta(\underset{\sim}{x})) \leq E_\theta\, \ell(\theta', \delta(\underset{\sim}{x})), \quad \forall \theta, \theta' \in \Theta$. for this particular loss-fct.

$\perp$.

#1.3.12.

Claim: $E_\theta[l(\theta, \delta_{r,s}(\underset{\sim}{X}))] \le E_\theta[l(\theta', \delta_{r,s}(\underset{\sim}{X}))]$, $\forall \theta, \theta' \in \Theta$.

First, If θ and θ' have the same sign, $E_\theta[l(\theta, \delta_{r,s}(\underset{\sim}{X}))] = E_\theta[l(\theta', \delta(\underset{\sim}{X}))]$.

Thus, it is enough to consider the six cases as follows.

Case 1) $\theta = 0$, $\theta' < 0$.

$$E_\theta[l(\theta', \delta_{r,s}(\underset{\sim}{X}))] - E_\theta[l(\theta, \delta_{r,s}(\underset{\sim}{X}))] = c\left[1 - \Phi\left(-z_{\frac{b}{b+c}}\right)\right] + b\left[1 - \Phi\left(z_{\frac{b}{b+c}}\right)\right]$$
$$- b\left[\Phi\left(-z_{\frac{b}{b+c}}\right) + 1 - \Phi\left(z_{\frac{b}{b+c}}\right)\right]$$
$$= c\left[1 - \Phi\left(-z_{\frac{b}{b+c}}\right)\right] - b\,\Phi\left(-z_{\frac{b}{b+c}}\right)$$
$$= c\,\Phi\left(z_{\frac{b}{b+c}}\right) - b\left(1 - \Phi\left(z_{\frac{b}{b+c}}\right)\right) = (b+c)\frac{b}{b+c} - b = 0$$

Case 2) $\theta = 0$, $\theta' > 0$.

$$E_\theta[l(\theta', \delta_{r,s}(\underset{\sim}{X}))] - E_\theta[l(\theta, \delta_{r,s}(\underset{\sim}{X}))] = b\,\Phi\left(-z_{\frac{b}{b+c}}\right) + c\,\Phi\left(z_{\frac{b}{b+c}}\right)$$
$$- b\left[\Phi\left(-z_{\frac{b}{b+c}}\right) + 1 - \Phi\left(z_{\frac{b}{b+c}}\right)\right] = (c+b)\cdot\frac{b}{b+c} - b = 0.$$

Case 3) $\theta < 0$, $\theta' > 0$.

$$E_\theta[l(\theta', \delta_{r,s}(\underset{\sim}{X}))] - E_\theta[l(\theta, \delta_{r,s}(\underset{\sim}{X}))] = b\,\Phi\left(-z_{\frac{b}{b+c}} - \sqrt{n}\theta\right) + c\,\Phi\left(z_{\frac{b}{b+c}} - \sqrt{n}\theta\right)$$
$$- c\left[1 - \Phi\left(-z_{\frac{b}{b+c}} - \sqrt{n}\theta\right)\right] - b\left[1 - \Phi\left(z_{\frac{b}{b+c}} - \sqrt{n}\theta\right)\right]$$
$$= (b+c)\,\Phi\left(-z_{\frac{b}{b+c}} - \sqrt{n}\theta\right) + (c+b)\,\Phi\left(z_{\frac{b}{b+c}} - \sqrt{n}\theta\right) - (b+c)$$
$$= -(b+c)\,\Phi\left(z_{\frac{b}{b+c}} + \sqrt{n}\theta\right) + (b+c)\,\Phi\left(z_{\frac{b}{b+c}} - \sqrt{n}\theta\right)$$
$$= (b+c)\left[\Phi\left(z_{\frac{b}{b+c}} - \sqrt{n}\theta\right) - \Phi\left(z_{\frac{b}{b+c}} + \sqrt{n}\theta\right)\right] > 0$$

$\because$ $z_{\frac{b}{b+c}} - \sqrt{n}\theta > z_{\frac{b}{b+c}} + \sqrt{n}\theta$ for $\theta < 0$ and Φ is nondecreasing.

Case 4) $\theta > 0,\ \theta' < 0$

$$E_\theta[l(\theta', \delta_{r,s}(\underset{\sim}{X}))] - E_\theta[l(\theta, \delta_{r,s}(\underset{\sim}{X}))] = c\left[1 - \Phi\left(-z\frac{b}{b+c} - \sqrt{n}\theta\right)\right] + b\left[1 - \Phi\left(z\frac{b}{b+c} - \sqrt{n}\theta\right)\right]$$

$$- b\,\Phi\left(-z\frac{b}{b+c} - \sqrt{n}\theta\right) - c\,\Phi\left(z\frac{b}{b+c} - \sqrt{n}\theta\right)$$

$$= (b+c)\,\Phi\left(z\frac{b}{b+c} + \sqrt{n}\theta\right) - (b+c)\,\Phi\left(z\frac{b}{b+c} - \sqrt{n}\theta\right)$$

$\because\ z\frac{b}{b+c} + \sqrt{n}\theta > z\frac{b}{b+c} - \sqrt{n}\theta$ for $\theta > 0$ and Φ is nondecreasing.

$$= (b+c)\left[\Phi\left(z\frac{b}{b+c} + \sqrt{n}\theta\right) - \Phi\left(z\frac{b}{b+c} - \sqrt{n}\theta\right)\right] > 0$$

Case 5) $\theta > 0,\ \theta' = 0$

$$E_\theta[l(\theta', \delta_{r,s}(\underset{\sim}{X}))] - E_\theta[l(\theta, \delta_{r,s}(\underset{\sim}{X}))] = b\,\Phi\left(-z\frac{b}{b+c} - \sqrt{n}\theta\right) + 1 - \Phi\left(z\frac{b}{b+c} - \sqrt{n}\theta\right)]$$

$$- b\,\Phi\left(-z\frac{b}{b+c} - \sqrt{n}\theta\right) - c\,\Phi\left(z\frac{b}{b+c} - \sqrt{n}\theta\right)$$

$$= b - (b+c)\,\Phi\left(z\frac{b}{b+c} - \sqrt{n}\theta\right) > b - (b+c)\,\Phi\left(z\frac{b}{b+c}\right)$$

$$= 0$$

(Since $-\Phi\left(z\frac{b}{b+c} - \sqrt{n}\theta\right) > -\Phi\left(z\frac{b}{b+c}\right)$ for $\theta > 0$).

Case 6) $\theta < 0,\ \theta' = 0$

$$E_\theta[l(\theta', \delta_{r,s}(\underset{\sim}{X}))] - E_\theta[l(\theta, \delta_{r,s}(\underset{\sim}{X}))] = b\left[\Phi\left(-z\frac{b}{b+c} - \sqrt{n}\theta\right) + 1 - \Phi\left(z\frac{b}{b+c} - \sqrt{n}\theta\right)\right]$$

$$- c\left[1 - \Phi\left(-z\frac{b}{b+c} - \sqrt{n}\theta\right)\right] - b\left[1 - \Phi\left(z\frac{b}{b+c} - \sqrt{n}\theta\right)\right]$$

$$= (b+c)\,\Phi\left(-z\frac{b}{b+c} - \sqrt{n}\theta\right) - c > (b+c)\,\Phi\left(-z\frac{b}{b+c}\right) - c$$

$$= (b+c)\cdot\frac{c}{b+c} - c = 0$$

(Since $\Phi\left(-z\frac{b}{b+c} - \sqrt{n}\theta\right) > \Phi\left(-z\frac{b}{b+c}\right)$ for $\theta < 0$) //

#1.5.1. (a) For $t \in \{0,1,2,\cdots\}$,

$$P_\theta(X_1=x_1, X_2=x_2, \cdots, X_n=x_n \mid \sum_{i=1}^{n} X_i = t) = \frac{P_\theta(X_1=x_1,\cdots,X_n=x_n, \sum_{i=1}^{n} X_i = t)}{P_\theta(\sum_{i=1}^{n} X_i = t)}$$

$$= \frac{\left[\prod_{i=1}^{n}\frac{1}{x_i!}\right] e^{-n\theta}\cdot\theta^{t}\cdot\prod_{i=1}^{n} I_{[x_i\in\mathcal{X}]}}{\frac{e^{-n\theta}(n\theta)^t}{t!}} \quad \text{as } \sum_{i=1}^{n} X_i \sim Po(n\theta)$$

$$= \frac{\left[\prod_{i=1}^{n}\frac{1}{x_i!}\right] t!\ \prod_{i=1}^{n} I_{[x_i\in\mathcal{X}]}}{n^t} \quad \text{where } \mathcal{X}=\{0,1,2,\cdots\}$$

which is indep. of θ. Thus, $\sum_{i=1}^{n} X_i$ is sufficient for θ. //

(b) $$P_\theta(X_1=x_1,\cdots,X_n=x_n) = \prod_{i=1}^{n}\frac{e^{-\theta}\cdot\theta^{x_i}}{x_i!} I_{[x_i\in\mathcal{X}]} = e^{-n\theta}\cdot\theta^{\sum_1^n x_i}\cdot\left[\prod_{i=1}^{n}\frac{1}{x_i!}\cdot I_{[x_i\in\mathcal{X}]}\right]$$

where $\mathcal{X}=\{0,1,2,\cdots\}$.

Let $g(t,\theta)=e^{-n\theta}\cdot\theta^t$ and $h(x)=\prod_{i=1}^{n}\frac{1}{x_i!} I_{[x_i\in\mathcal{X}]}$

$\Rightarrow$ $P_\theta(X_1=x_1,\cdots,X_n=x_n) = g(T(x),\theta)\cdot h(x)$ where $T(x)=\sum_1^n x_i$

By the factorization thm, $T(X)=\sum_{i=1}^{n} X_i$ is sufficient for θ. //

1.5.2

i) Direct approach: For t s.t. $\max\{0, n-N(1-\theta)\} \le t \le \min\{n, N\theta\}$,

$$P_\theta(X_1 = x_1, \cdots, X_n = x_n \mid \sum_1^n X_i = t) = \frac{1}{\binom{n}{t}} \quad \text{which is free of } \theta.$$

Thus, $T(\underset{\sim}{X}) = \sum_1^n X_i$ is sufficient for θ. //

ii) Factorization thm:

$$P_\theta(X_1 = x_1, \cdots, X_n = x_n) = \frac{\binom{N\theta}{\sum_1^n x_i}\binom{N-N\theta}{n-\sum_1^n x_i}}{\binom{N}{n}} \cdot \frac{1}{\binom{n}{\sum_1^n x_i}},$$

where $\sum_1^n x_i \in (\max\{0, n-N+N\theta\}, \min\{n, N\theta\}) \equiv \mathcal{J}$

Let $g(t,\theta) = \dfrac{\binom{N\theta}{t}\binom{N-N\theta}{n-t}}{\binom{n}{t}} \cdot I_{[t \in \mathcal{J}]}$ and $h(\underset{\sim}{x}) = \dfrac{1}{\binom{N}{n}}$

Then,

$$P_\theta(X_1 = x_1, \cdots, X_n = x_n) = g(T(\underset{\sim}{x}), \theta) \cdot h(\underset{\sim}{x}) \quad \text{where } T(\underset{\sim}{X}) = \sum_1^n X_i.$$

By the factorization thm, $T(\underset{\sim}{X}) = \sum_1^n X_i$ is sufficient for θ. //

#1.5.7 (a)

$$f(\underset{\sim}{x}|\theta) = \prod_{i=1}^{n} \theta \cdot x_i^{\theta-1} \cdot I_{(x_i \in (0,1))} = \theta^n \left[\prod_{i=1}^{n} x_i\right]^{\theta-1} \cdot \prod_{i=1}^{n} I_{[x_i \in (0,1)]}$$

Let $g(t,\theta) = \theta^n \cdot t^{\theta-1}$ and $h(\underset{\sim}{x}) = \prod_{i=1}^{n} I_{[x_i \in (0,1)]}$

Then, $f(\underset{\sim}{x}|\theta) = g(T(\underset{\sim}{x}), \theta) \cdot h(\underset{\sim}{x})$ where $T(\underset{\sim}{x}) = \prod_{i=1}^{n} x_i$

By the factorization thm, $T(\underset{\sim}{x}) = \prod_{i=1}^{n} x_i$ is sufficient for θ. //

(b)

For any fixed $a>0$,

$$f(x|\theta) = \prod_{i=1}^{n} \theta \cdot a \cdot x_i^{a-1} \exp(-\theta x_i^a)\, I_{[x_i>1]} = \theta^n a^n \left[\prod_1^n x_i\right]^{a-1} \exp\left(-\theta \sum_1^n x_i^a\right) \prod_1^n I_{[x_i>1]}$$

Let $g(t,\theta) = \theta^n \exp(-\theta t)$ and $h(\underset{\sim}{x}) = a^n \left[\prod_1^n x_i\right]^{a-1} \cdot \prod_1^n I_{[x_i>1]}$

Then, $f(\underset{\sim}{x}|\theta) = g(T(\underset{\sim}{x}), \theta) \cdot h(\underset{\sim}{x})$ where $T(\underset{\sim}{x}) = \sum_1^n x_i^a$

By the factorization thm, $T(\underset{\sim}{x}) = \sum_1^n x_i^a$ is sufficient for θ when a is fixed. //

(c)

For any fixed $a>0$,

$$f(\underset{\sim}{x}|\theta) = \prod_{i=1}^{n} \frac{\theta a^\theta}{x_i^{\theta+1}} I_{[x_i>a]} = \theta^n a^{n\theta} \left[\prod_1^n x_i\right]^{-(\theta+1)} \prod_1^n I_{[x_i>a]}$$

Let $g(t,\theta) = \theta^n a^{n\theta} t^{-(\theta+1)}$ and $h(\underset{\sim}{x}) = \prod_1^n I_{[x_i>a]}$.

Then, $f(\underset{\sim}{x}|\theta) = g(T(\underset{\sim}{x}), \theta) \cdot h(\underset{\sim}{x})$ where $T(\underset{\sim}{x}) = \prod_1^n x_i$

By the factorization thm, $T(\underset{\sim}{x}) = \prod_1^n x_i$ is sufficient for θ when a is fixed. //

#1.5.7. (a)

For any fixed σ,

$$f(\underline{x}\mid\mu) = \prod_{i=1}^{n} \frac{1}{\sigma} \exp\left\{-\frac{x_i-\mu}{\sigma}\right\} \cdot I_{[x_i \geq \mu]} = \sigma^{-n} \exp\left\{-\frac{\sum_1^n (x_i-\mu)}{\sigma}\right\} \cdot I_{[x_{(1)} \geq \mu]}$$

$$= \sigma^{-n} \exp\left(-\frac{\sum_1^n x_i}{\sigma}\right) \cdot \exp\left(\frac{n\mu}{\sigma}\right) \cdot I_{[x_{(1)} \geq \mu]}$$

where $x_{(1)} = \min_i x_i$.

Let $g(t,\mu) = \exp\left(\frac{n\mu}{\sigma}\right) \cdot I_{[x_{(1)} \geq \mu]}$ and $h(\underline{x}) = \sigma^{-n} \exp\left(-\frac{\sum_1^n x_i}{\sigma}\right)$.

Then, $f(\underline{x}\mid\mu) = g(T(\underline{x}),\mu) \cdot h(\underline{x})$ where $T(\underline{x}) = x_{(1)}$.

By the factorization thm, $T(\underline{X}) = X_{(1)}$ is sufficient for μ when σ is fixed. //

(b)

For any fixed μ,

$$f(\underline{x}\mid\sigma) = \sigma^{-n} \exp\left\{-\frac{\sum_1^n (x_i-\mu)}{\sigma}\right\} \cdot I_{[x_{(1)} \geq \mu]} = \sigma^{-n} \exp\left\{-\frac{\sum_1^n x_i - n\mu}{\sigma}\right\} \cdot I_{[x_{(1)} \geq \mu]}$$

Let $g(t,\sigma) = \sigma^{-n} \exp\left\{-\frac{t-n\mu}{\sigma}\right\}$ and $h(\underline{x}) = I_{[x_{(1)} \geq \mu]}$

Then, $f(\underline{x}\mid\sigma) = g(T(\underline{x}),\sigma) \cdot h(\underline{x})$ where $T(\underline{x}) = \sum_1^n x_i$.

By the factorization thm, $T(\underline{X}) = \sum_1^n X_i$ is sufficient for σ when μ is fixed. //

(c)

$$f(\underline{x}\mid\mu,\sigma) = \sigma^{-n} \exp\left\{-\frac{\sum_1^n x_i - n\mu}{\sigma}\right\} \cdot I_{[x_{(1)} \geq \mu]}$$

Let $g(\underline{t},\mu,\sigma) = \sigma^{-n} \exp\left\{-\frac{t_2-n\mu}{\sigma}\right\} \cdot I_{[t_1 \geq \mu]}$ where $\underline{t} = (t_1,t_2)$ and $h(\underline{x}) = 1$.

Then, $f(\underline{x}\mid\mu,\sigma) = g(T(\underline{x}),\mu,\sigma) \cdot h(\underline{x})$ where $T(\underline{x}) = (x_{(1)}, \sum_1^n x_i)$.

By the factorization thm, $T(\underline{X}) = (X_{(1)}, \sum_1^n X_i)$ is sufficient for (μ,σ). //

#1.5.9.

i) $f_{\underset{\sim}{\theta}}(\underset{\sim}{x}) = \prod_{i=1}^{n} a(\underset{\sim}{\theta}) h(x_i) \cdot I_{[\theta_1 \le x_i \le \theta_2]} = a(\underset{\sim}{\theta})^n \left[\prod_{i=1}^{n} h(x_i)\right] I_{[\theta_1 \le x_{(1)}]} \cdot I_{[x_{(n)} \le \theta_2]}$

where $x_{(1)} = \min_i x_i$ and $x_{(n)} = \max_i x_i$.

Let $g(\underset{\sim}{t}, \underset{\sim}{\theta}) = a(\underset{\sim}{\theta})^n \cdot I_{[\theta_1 \le t_1]} \cdot I_{[t_2 \le \theta_2]}$ where $\underset{\sim}{t} = (t_1, t_2)$. and $c(\underset{\sim}{x}) = \prod_{1}^{n} h(x_i)$

Then $f_{\underset{\sim}{\theta}}(\underset{\sim}{x}) = g(T(\underset{\sim}{x}), \underset{\sim}{\theta}) \cdot c(\underset{\sim}{x})$ where $T(\underset{\sim}{x}) = (x_{(1)}, x_{(n)})$.

By the factorization thm, $T(\underset{\sim}{x}) = (x_{(1)}, x_{(n)})$ is sufficient for $\underset{\sim}{\theta} = (\theta_1, \theta_2)$.

ii) For $U_{[\theta_1, \theta_2]}$, $a(\underset{\sim}{\theta}) = \frac{1}{\theta_2 - \theta_1}$ and $h(x) = 1$.

$\Rightarrow$ $T(\underset{\sim}{x}) = (x_{(1)}, x_{(n)})$ is sufficient for $\underset{\sim}{\theta} = (\theta_1, \theta_2)$ //

#1.5.11

i) $f(\underset{\sim}{x}|\theta) = \prod_{i=1}^{n} \frac{1}{\theta} I_{[0 < x_i < \theta]} = \frac{1}{\theta^n} I_{[0 < x_{(1)}]} \cdot I_{[x_{(n)} < \theta]}$ where $x_{(1)} = \min_i x_i$, $x_{(n)} = \max_i x_i$.

Let $g(t, \theta) = \frac{1}{\theta^n} \cdot I_{[t < \theta]}$ and $h(\underset{\sim}{x}) = I_{[0 < x_{(1)}]}$ $\Rightarrow$ $f(\underset{\sim}{x}|\theta) = g(T(\underset{\sim}{x}), \theta) \cdot h(\underset{\sim}{x})$ where $T(\underset{\sim}{x}) = x_{(n)}$.

By the factorization thm, $T(\underset{\sim}{x}) = x_{(n)}$ is sufficient for θ.

ii) Suppose that $\exists\ k(\underset{\sim}{x}, \underset{\sim}{y}) > 0$ s.t. $f(\underset{\sim}{y}|\theta) = k(\underset{\sim}{x}, \underset{\sim}{y}) f(\underset{\sim}{x}|\theta)$ for $\forall \theta$,

i.e., $\frac{1}{\theta^n} I_{[0 < y_{(1)}]} \cdot I_{[y_{(n)} < \theta]} = k(\underset{\sim}{x}, \underset{\sim}{y}) \cdot \frac{1}{\theta^n} I_{[0 < x_{(1)}]} \cdot I_{[x_{(n)} < \theta]}$, $\forall \theta$.

$\Leftrightarrow$ $x_{(n)} = y_{(n)}$ holds

$\because$ $\Leftarrow$: clear.

$\Rightarrow$: Let $x_{(n)} < y_{(n)}$. Then, for $\theta \in (x_{(n)}, y_{(n)})$, LHS $= 0$ but RHS $= k(\underset{\sim}{x}, \underset{\sim}{y}) \cdot \frac{1}{\theta^n} > 0$ $\ast$.

Let $x_{(n)} > y_{(n)}$. Then, for $\theta \in (y_{(n)}, x_{(n)})$, LHS $= \frac{1}{\theta^n} > 0$ but RHS $= 0$. $\ast$

Thus, $\exists\ k(x, y) > 0$ s.t. $f(y|\theta) = k(x, y) \cdot f(x|\theta)$ for $\forall \theta$

Iff $X_{(n)} = Y_{(n)}$.

By i) and ii), $T(X) = X_{(n)}$ is minimal sufficient for θ. //

Problem 4.

Note that for $S(x) = \sum_{\theta=1}^{4} f(x|\theta)$, $x \in \{1, 2, \cdots, 14\}$,

$$l(x) = \left(\frac{f(x|1)}{S(x)}, \frac{f(x|2)}{S(x)}, \frac{f(x|3)}{S(x)}, \frac{f(x|4)}{S(x)} \right)$$ is minimal sufficient for θ.

	x: (1	2	3	4	5	6	7	8	9	10	11	12	13	14)
$\frac{f(x\|4)}{S(x)}$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	0	$\frac{1}{6}$	$\frac{2}{7}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{6}$
$\frac{f(x\|3)}{S(x)}$	0	$\frac{1}{3}$	0	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{5}{7}$	$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{1}{4}$	0	$\frac{1}{6}$	0	$\frac{1}{6}$
$\frac{f(x\|2)}{S(x)}$	0	$\frac{1}{6}$	$\frac{2}{3}$	0	$\frac{1}{6}$	0	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$
$\frac{f(x\|1)}{S(x)}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{2}$	$\frac{1}{3}$	0	$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{1}{4}$	1	$\frac{1}{3}$	0	$\frac{1}{3}$

Thus, $l(x)$ s.t.

$l(1) = (\frac{2}{3}, 0, 0, \frac{1}{3})$,

$l(2) = l(5) = (\frac{1}{6}, \frac{1}{3}, \frac{1}{6}, \frac{1}{3})$

$l(3) = (\frac{1}{3}, 0, \frac{2}{3}, 0)$

$l(4) = l(7) = (0, \frac{1}{2}, 0, \frac{1}{2})$ is minimal sufficient for θ //

$l(6) = (\frac{2}{7}, \frac{5}{7}, 0, 0)$

$l(8) = l(10) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.

$l(9) = l(13) = (\frac{1}{2}, 0, \frac{1}{2}, 0)$

$l(11) = (0, 0, 0, 1)$

$l(12) = l(14) = (\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3})$

Problem 5.

1.5.5 (a)

Suppose that ① $T_1(\underset{\sim}{X})$ is sufficient for θ_1 when θ_2 is fixed

② $T_2(\underset{\sim}{X})$ is sufficient for θ_2 when θ_1 is fixed

③ θ_1 and θ_2 vary independently, $\theta_1 \in \Theta_1$ and $\theta_2 \in \Theta_2$

④ T_1 and T_2 do not dep. on θ_2 and θ_1, respectively

<u>claim:</u> $(T_1(\underset{\sim}{x}), T_2(\underset{\sim}{x}))$ is sufficient for $\underset{\sim}{\theta} = (\theta_1, \theta_2)$

i) For any fixed $\theta_2 \in \Theta_2$, $f(\underset{\sim}{x} | \theta_1, \theta_2) = g(T_1(\underset{\sim}{x}), \theta_1, \theta_2) \cdot h(\underset{\sim}{x}, \theta_2)$ by ① & factorization thm.

By ④, $g(T_1(\underset{\sim}{x}), \theta_1, \theta_2) = g_1(T_1(\underset{\sim}{x}), \theta_1) \cdot c(\theta_1, \theta_2)$

$\Rightarrow$ $f(\underset{\sim}{x} | \theta_1, \theta_2) = g_1(T_1(\underset{\sim}{x}), \theta_1) \cdot c(\theta_1, \theta_2) \cdot h(\underset{\sim}{x}, \theta_2)$, $\forall \theta_1 \in \Theta_1$, $\theta_2 \in \Theta_2$.

ii) For any fixed $\theta_1 \in \Theta_1$, $f(\underset{\sim}{x} | \theta_1, \theta_2) = g_1(T(\underset{\sim}{x}), \theta_1) \cdot c(\theta_1, \theta_2) \cdot h(\underset{\sim}{x}, \theta_2)$ where

$h(\underset{\sim}{x}, \theta_2) = h_1(T_2(\underset{\sim}{x}), \theta_2) \cdot h_2(\underset{\sim}{x})$, by ② + factorization thm.

Then, we can have

$$f(\underset{\sim}{x} | \theta_1, \theta_2) = \underbrace{g_1(T_1(\underset{\sim}{x}), \theta_1) \cdot c(\theta_1, \theta_2) \cdot h_1(T_2(\underset{\sim}{x}), \theta_2)}_{\equiv} \cdot h_2(\underset{\sim}{x})$$

$$\overset{\text{let}}{=} \tilde{g}(T_1(\underset{\sim}{x}), T_2(\underset{\sim}{x}), \theta_1, \theta_2) \cdot h_2(\underset{\sim}{x}).$$

By the factorization thm, $T(\underset{\sim}{X}) = (T_1(\underset{\sim}{X}), T_2(\underset{\sim}{X}))$ is sufficient for $\underset{\sim}{\theta} = (\theta_1, \theta_2)$.

//

(b)

i) Consider $X_1, \cdots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, where $-\infty < \mu < \infty$, $\sigma^2 > 0$ and μ, σ^2: unknown.

$$\Rightarrow \quad f(\underset{\sim}{x} \mid \mu, \sigma^2) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{ -\frac{1}{2\sigma^2} \sum_1^n x_i^2 + \frac{\mu}{\sigma^2} \sum_1^n x_i - \frac{n\mu}{2\sigma^2} \right\}$$

Let $g(t_1, t_2, \mu, \sigma^2) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{ -\frac{1}{2\sigma^2} t_2 + \frac{\mu}{\sigma^2} t_1 - \frac{n\mu}{2\sigma^2} \right\}$ and $h(\underset{\sim}{x}) = 1$.

Then, $f(\underset{\sim}{x} \mid \mu, \sigma^2) = g(T_1(\underset{\sim}{x}), T_2(\underset{\sim}{x}), \mu, \sigma^2) \cdot h(\underset{\sim}{x})$ where $T_1(\underset{\sim}{x}) = \sum_1^n x_i$, $T_2(\underset{\sim}{x}) = \sum_1^n x_i^2$.

By the factorization thm, $T(\underset{\sim}{X}) = (\sum_1^n X_i, \sum_1^n X_i^2)$ is sufficient for (μ, σ^2).

ii) When σ^2 is fixed, $T_1(\underset{\sim}{X}) = \sum_1^n X_i$ is sufficient for μ

(by the factorization thm with $g(T_1(\underset{\sim}{x}), \mu) = \exp\left\{ \frac{\mu}{\sigma^2} T_1(\underset{\sim}{x}) - \frac{n\mu}{2\sigma^2} \right\}$,

$h(\underset{\sim}{x}) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{ -\frac{1}{2\sigma^2} \sum_1^n x_i^2 \right\}$, and

$T(\underset{\sim}{x}) = \sum_1^n x_i$).

But, when μ is fixed, $T_2(\underset{\sim}{X}) = \sum_1^n X_i^2$ is not sufficient for σ^2.

(For any fixed μ, we cannot find fcts of $g(T_2(\underset{\sim}{x}), \sigma^2)$ and $h(\underset{\sim}{x})$

s.t. $f(\underset{\sim}{x} \mid \sigma^2) = g(T_2(\underset{\sim}{x}), \sigma^2) \cdot h(\underset{\sim}{x})$ because of the term of $\exp\left(\frac{\mu}{\sigma^2} \sum_1^n x_i \right)$)

//

4.5.16 (a)

$$f(\underset{\sim}{x}, \underset{\sim}{y} \mid \mu, \eta, \sigma, \tau) = (2\pi\sigma^2)^{-\frac{m}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_1^m (x_i-\mu)^2\right\} \cdot (2\pi\tau^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\tau^2}\sum_1^n (y_i-\eta)^2\right\}.$$

$$= (2\pi)^{-\frac{m+n}{2}} (\sigma^2)^{-\frac{m}{2}} (\tau^2)^{-\frac{n}{2}} \exp\left\{-\frac{m\mu^2}{2\sigma^2} - \frac{n\eta^2}{2\tau^2}\right\} \exp\left\{-\frac{\sum_1^m x_i^2}{2\sigma^2} + \frac{\mu\sum_1^m x_i}{\sigma^2} - \frac{\sum_1^n y_i^2}{2\tau^2} + \frac{\eta\sum_1^n y_i}{\tau^2}\right\}$$

For $\underset{\sim}{\theta} = (\mu, \eta, \sigma, \tau)$,

let $\eta(\underset{\sim}{\theta}) = \left(-\frac{1}{2\sigma^2}, \frac{\mu}{\sigma^2}, -\frac{1}{2\tau^2}, \frac{\eta}{\tau^2}\right)$ and $T(\underset{\sim}{X}, \underset{\sim}{Y}) = \left(\sum_1^m X_i^2, \sum_1^m X_i, \sum_1^n Y_i^2, \sum_1^n Y_i\right)$.

For $\Theta = \{(\mu, \eta, \sigma, \tau) : \mu \in \mathbb{R}^1, \eta \in \mathbb{R}^1, \sigma^2 > 0, \tau^2 > 0\}$,

$\mathcal{E}_\Theta = \{\eta(\underset{\sim}{\theta}) \equiv (\eta_1, \eta_2, \eta_3, \eta_4) : \eta_1 < 0, \eta_2 \in \mathbb{R}^1, \eta_3 < 0, \eta_4 \in \mathbb{R}^1\}$ contains an open rectangle.

By the property (thm) of exponential family, $T(\underset{\sim}{X}, \underset{\sim}{Y})$ is minimal sufficient for $\underset{\sim}{\theta}$. //

(b)

$$f(\underset{\sim}{x}, \underset{\sim}{y} \mid \mu, \eta, \sigma) = (2\pi\sigma^2)^{-\frac{m+n}{2}} \exp\left\{-\frac{m\mu^2+n\eta^2}{2\sigma^2}\right\} \exp\left\{-\frac{\sum_1^m x_i^2 + \sum_1^n y_i^2}{2\sigma^2} + \frac{\mu \cdot \sum_1^m x_i}{\sigma^2} + \frac{\eta\sum_1^n y_i}{\sigma^2}\right\}$$

For $\underset{\sim}{\theta} = (\mu, \eta, \sigma)$,

let $\eta(\underset{\sim}{\theta}) = \left(-\frac{1}{2\sigma^2}, \frac{\mu}{\sigma^2}, \frac{\eta}{\sigma^2}\right)$ and $T(\underset{\sim}{X}, \underset{\sim}{Y}) = \left(\sum_1^m X_i^2 + \sum_1^n Y_i^2, \sum_1^m X_i, \sum_1^n Y_i\right)$.

For $\Theta = \{(\mu, \eta, \sigma) : \mu \in \mathbb{R}^1, \eta \in \mathbb{R}^1, \sigma^2 > 0\}$,

$\mathcal{E}_\Theta = \{\eta(\underset{\sim}{\theta}) = (\eta_1, \eta_2, \eta_3) : \eta_1 < 0, \eta_2 \in \mathbb{R}^1, \eta_3 \in \mathbb{R}^1\}$ contains an open rectangle.

By the property (thm) of exponential family, $T(\underset{\sim}{X}, \underset{\sim}{Y})$ is minimal sufficient for $\underset{\sim}{\theta}$.

//

(c)

i) $$f(\underset{\sim}{x}, \underset{\sim}{y} \mid \mu, \sigma, \tau) = (2\pi)^{-\frac{m+n}{2}} (\sigma^2)^{-\frac{m}{2}} (\tau^2)^{-\frac{n}{2}} \exp\left\{-\left(\frac{m}{2\sigma^2} + \frac{n}{2\tau^2}\right)\mu^2\right\} \times$$

$$\exp\left\{-\frac{\sum_1^m x_i^2}{2\sigma^2} + \frac{\mu \sum_1^m x_i}{\sigma^2} - \frac{\sum_1^n y_i^2}{2\tau^2} + \frac{\mu \sum_1^n y_i}{\tau^2}\right\}$$

Let $g(t_1, t_2, t_3, t_4, \underset{\sim}{\theta}) = (\sigma^2)^{-\frac{m}{2}} (\tau^2)^{-\frac{n}{2}} \exp\left\{-\left(\frac{m}{2\sigma^2} + \frac{n}{2\tau^2}\right)\mu^2\right\} \cdot \exp\left\{-\frac{t_1}{2\sigma^2} + \frac{\mu t_2}{\sigma^2} - \frac{t_3}{2\tau^2} + \frac{\mu t_4}{\tau^2}\right\}$

and $h(\underset{\sim}{x}, \underset{\sim}{y}) = 1$. where $\underset{\sim}{\theta} = (\mu, \sigma, \tau)$.

$\Rightarrow$ $f(\underset{\sim}{x}, \underset{\sim}{y} \mid \underset{\sim}{\theta}) = g(T_1(\underset{\sim}{x}), T_2(\underset{\sim}{x}), T_3(\underset{\sim}{x}), T_4(\underset{\sim}{x}), \underset{\sim}{\theta}) \cdot h(\underset{\sim}{x}, \underset{\sim}{y})$

where $(T_1(\underset{\sim}{x}), T_2(\underset{\sim}{x}), T_3(\underset{\sim}{y}), T_4(\underset{\sim}{y})) = \left(\sum_1^m x_i^2, \sum_1^m x_i, \sum_1^n y_i^2, \sum_1^n y_i\right) \equiv T(\underset{\sim}{x}, \underset{\sim}{y})$

By the factorization thm, $T(\underset{\sim}{x}, \underset{\sim}{y})$ is sufficient for $\underset{\sim}{\theta} = (\mu, \sigma, \tau)$.

ii) WLOG, for $\underset{\sim}{d}_1 = (x_{11}, \cdots, x_{1m}, y_{11}, \cdots, y_{1n})$ and $\underset{\sim}{d}_2 = (x_{21}, \cdots, x_{2m}, y_{21}, \cdots, y_{2n})$,

assume that $\sum_1^m x_{1i}^2 \geq \sum_1^m x_{2i}^2$, $\sum_1^m x_{1i} \geq \sum_1^m x_{2i}$, $\sum_1^n y_{1i}^2 \geq \sum_1^n y_{2i}^2$ and $\sum_1^n y_{1i} \geq \sum_1^n y_{2i}$.

Suppose that $\exists\ k(\underset{\sim}{d}_1, \underset{\sim}{d}_2) > 0$ s.t.

$$f(\underset{\sim}{d}_1 \mid \underset{\sim}{\theta}) = k(\underset{\sim}{d}_1, \underset{\sim}{d}_2) \cdot f(\underset{\sim}{d}_2 \mid \underset{\sim}{\theta}) \quad \text{for } \forall \underset{\sim}{\theta}$$

i.e., $$\frac{f(\underset{\sim}{d}_1 \mid \underset{\sim}{\theta})}{f(\underset{\sim}{d}_2 \mid \underset{\sim}{\theta})} = \exp\left[-\frac{\sum_1^m x_{1i}^2 - \sum_1^m x_{2i}^2}{2\sigma^2} - \frac{\sum_1^n y_{1i}^2 - \sum_1^n y_{2i}^2}{2\tau^2} + \frac{\mu}{\sigma^2}\left(\sum_1^m x_{1i} - \sum_1^m x_{2i}\right) + \frac{\mu}{\tau^2}\left(\sum_1^n y_{1i} - \sum_1^n y_{2i}\right)\right]$$

for all $\underset{\sim}{\theta}$.

iff $T(\underset{\sim}{x}_1, \underset{\sim}{y}_1) = T(\underset{\sim}{x}_2, \underset{\sim}{y}_2)$

$\therefore$ $T(\underset{\sim}{x}, \underset{\sim}{y}) = \left(\sum_1^m x_i^2, \sum_1^m x_i, \sum_1^n y_i^2, \sum_1^n y_i\right)$ is minimal sufficient for $\underset{\sim}{\theta} = (\mu, \sigma, \tau)$.

//

#6. i) For any pair x, x',

suppose that $\ell(x) = \ell(x')$ i.e., $\frac{f(x|\theta)}{s(x)} = \frac{f(x'|\theta)}{s(x')}$, $\theta = 1, 2, \cdots, k$

where $s(x) = \sum_{\theta=1}^{k} f(x|\theta)$.

For any $\theta, \theta' \in \Theta$, $\frac{f(x|\theta)}{f(x|\theta')} = \frac{f(x|\theta)/s(x)}{f(x|\theta')/s(x)} = \frac{f(x'|\theta)/s(x')}{f(x'|\theta')/s(x')} = \frac{f(x'|\theta)}{f(x'|\theta')}$.

By the likelihood ratio criterion, $\ell(x)$ is sufficient for θ.

ii) <u>claim</u>: $\exists\ k(x,y) > 0$ s.t. $f(y|\theta) = k(x,y) f(x|\theta)$, $\forall \theta = 1, \cdots, k$

iff $\ell(x) = \ell(y)$.

$\Leftarrow$: Suppose that $\ell(x) = \ell(y)$. i.e., $\frac{f(x|\theta)}{s(x)} = \frac{f(y|\theta)}{s(y)}$, $\forall \theta$

Then, $f(y|\theta) = \frac{s(y)}{s(x)} \cdot f(x|\theta) = k(x,y) \cdot f(x|\theta)$, $\forall \theta$

where $k(x,y) = \frac{s(y)}{s(x)}$.

$\Rightarrow$: Suppose that $\exists\ k(x,y) > 0$ s.t. $f(y|\theta) = k(x,y) f(x|\theta)$, $\forall \theta = 1, \cdots, k$

Then, $s(y) = \sum_{\theta=1}^{k} f(y|\theta) = \sum_{\theta=1}^{k} k(x,y) f(x|\theta) = k(x,y) \sum_{\theta=1}^{k} f(x|\theta) = k(x,y)\, s(x)$.

$$\ell(y) = \left(\frac{f(y|1)}{s(y)}, \cdots, \frac{f(y|k)}{s(y)} \right) = \left(\frac{k(x,y) f(x|1)}{k(x,y)\, s(x)}, \cdots, \frac{k(x,y) f(x|k)}{k(x,y) f(x|k)} \right)$$

$$= \left(\frac{f(x|1)}{s(x)}, \cdots, \frac{f(x|k)}{s(x)} \right) = \ell(x)$$

$\therefore$ $\ell(x) = \left(\frac{f(x|1)}{s(x)}, \cdots, \frac{f(x|k)}{s(x)} \right)$ is minimal sufficient for θ.

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