# STAT 543

## HW 4 solution

## Spring 2016

#2.1.2 (a)

For $X \sim Exp(\lambda)$, $f(x|\lambda) = \lambda e^{-\lambda x}$, $x>0,\ \lambda>0$

$$EX = \int_0^\infty x \cdot \lambda e^{-\lambda x}\,dx = \lambda \cdot \int_0^\infty x e^{-\lambda x}\,dx = \lambda \cdot \Gamma(2)\left(\frac{1}{\lambda}\right)^2 \cdot \underbrace{\int_0^\infty \frac{1}{\Gamma(2)\left(\frac{1}{\lambda}\right)^2} x e^{-\lambda x}\,dx}_{=1} = \frac{1}{\lambda}$$

$$\Rightarrow \quad EX = \frac{1}{\lambda} \overset{set}{=} \bar{X}_n \qquad \therefore \hat{\lambda}_{MME} = \frac{1}{\bar{X}_n} \quad //$$

(b)

$$EX^2 = \int_0^\infty x^2 \cdot \lambda e^{-\lambda x}\,dx = \lambda\Gamma(3)\left(\frac{1}{\lambda}\right)^3 \cdot \underbrace{\int_0^\infty \frac{1}{\Gamma(3)\left(\frac{1}{\lambda}\right)^3} x^2 e^{-\lambda x}\,dx}_{=1} = \frac{2}{\lambda^2}$$

$$\Rightarrow \quad EX^2 = \frac{2}{\lambda^2} \overset{set}{=} \frac{1}{n}\sum_1^n X_i^2 \qquad \therefore \hat{\lambda}_{MME} = \sqrt{\frac{2n}{\sum_{i=1}^n X_i^2}} \quad //$$

(c)

Note that $Var(X) = EX^2 - (EX)^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$.

$$\Rightarrow \quad \frac{1}{\lambda^2} = EX^2 - (EX)^2 \overset{set}{=} \frac{1}{n}\sum_1^n X_i^2 - \left(\frac{1}{n}\sum_1^n X_i\right)^2$$

$$\therefore \quad \hat{\lambda}_{MME} = \frac{1}{\sqrt{\frac{1}{n}\sum_1^n X_i^2 - \left(\frac{1}{n}\sum_1^n X_i\right)^2}} \quad //$$

(d)

$$P(X_1 \geq 1) = \int_1^\infty \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_1^\infty = e^{-\lambda}.$$

$$\Rightarrow \quad \widehat{e^{-\lambda}} = e^{-\hat{\lambda}} = e^{-\bar{X}_n} \quad //$$

#2.1.3.

$$EX = \frac{\alpha_1}{\alpha_1+\alpha_2} \quad \text{and} \quad EX^2 = \mathrm{Var}X + (EX)^2 = \frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^2(\alpha_1+\alpha_2+1)} + \left(\frac{\alpha_1}{\alpha_1+\alpha_2}\right)^2 = \frac{\alpha_1(\alpha_1+1)}{(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+1)}$$

$$\Rightarrow \begin{cases} EX = \dfrac{\alpha_1}{\alpha_1+\alpha_2} \equiv \mu_1 & \cdots \text{①} \\ EX^2 = \dfrac{\alpha_1(\alpha_1+1)}{(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+1)} \equiv \mu_2 & \cdots \text{②} \end{cases}$$

$$\Rightarrow \alpha_1 = \frac{\mu_1}{1-\mu_1}\alpha_2 \quad \text{by ①}$$

From ②, $\quad \mu_1 \cdot \dfrac{\alpha_1+1}{\alpha_1+\alpha_2+1} = \mu_2$

$$\Rightarrow \mu_1(\alpha_1+1) = \mu_2(\alpha_1+\alpha_2+1) \Rightarrow (\mu_2-\mu_1)\alpha_1 + \mu_2\alpha_2 + \mu_2-\mu_1 = 0.$$

$$\Rightarrow \left[(\mu_2-\mu_1)\left(\frac{\mu_1}{1-\mu_1}\right) + \mu_2\right]\alpha_2 = \mu_1-\mu_2$$

$$\therefore \begin{cases} \alpha_2 = \dfrac{\mu_1-\mu_2}{(\mu_2-\mu_1)\dfrac{\mu_1}{1-\mu_1}+\mu_2} = \dfrac{(\mu_1-\mu_2)(1-\mu_1)}{(\mu_2-\mu_1)\mu_1 + \mu_2(1-\mu_1)} = \dfrac{(\mu_1-\mu_2)(1-\mu_1)}{\mu_2-\mu_1^2} \\ \alpha_1 = \dfrac{\mu_1}{1-\mu_1}\cdot\dfrac{(\mu_1-\mu_2)(1-\mu_1)}{(\mu_2-\mu_1)\mu_1+\mu_2(1-\mu_1)} = \dfrac{\mu_1(\mu_1-\mu_2)}{(\mu_2-\mu_1)\mu_1+\mu_2(1-\mu_1)} = \dfrac{(\mu_1-\mu_2)\mu_1}{\mu_2-\mu_1^2} \end{cases}$$

Now, we plug in $\bar{X}_n$ and $\frac{1}{n}\sum_1^n X_i^2$ for $\mu_1 (\equiv EX)$ and $\mu_2 (\equiv EX^2)$.

$$\hat{\alpha}_1 = \frac{(\bar{X}_n - \frac{1}{n}\sum_1^n X_i^2)\bar{X}_n}{\frac{1}{n}\sum_1^n X_i^2 - \bar{X}_n^2}, \qquad \hat{\alpha}_2 = \frac{(\bar{X}_n - \frac{1}{n}\sum_1^n X_i^2)(1-\bar{X}_n)}{\frac{1}{n}\sum_1^n X_i^2 - \bar{X}_n^2} \quad //$$

#2.2.10. (a)

$L(\theta) = f(\underline{x};\theta) = \theta^n e^{-\theta \sum_1^n x_i}$

$\Rightarrow \ell(\theta) \overset{let}{=} \log L(\theta) = n \log \theta - \theta \sum_1^n x_i$

$\Rightarrow \ell'(\theta) = \frac{n}{\theta} - \sum_1^n x_i \overset{set}{=} 0 \qquad \Rightarrow \hat{\theta} = \frac{n}{\sum_1^n x_i}$

Since $\ell''(\theta) = -\frac{n}{\theta^2} < 0$ for $\forall \theta > 0$, $\hat{\theta}_{MLE} = \frac{n}{\sum_1^n x_i}$ //

(b)

$L(\theta) = f(\underline{x};\theta) = \theta^n c^{n\theta} \left[\prod_{i=1}^n x_i\right]^{-(\theta+1)}$

$\Rightarrow \ell(\theta) \overset{let}{=} \log L(\theta) = n \log \theta + n\theta \log c - (\theta+1) \sum_1^n \log x_i$

$\Rightarrow \ell'(\theta) = \frac{n}{\theta} + n \log c - \sum_1^n \log x_i \overset{set}{=} 0 \qquad \Rightarrow \hat{\theta} = \frac{n}{\sum_1^n \log x_i - n \log c}$

Since $\ell''(\theta) = -\frac{n}{\theta^2} < 0$ for $\forall \theta > 0$, $\hat{\theta}_{MLE} = \frac{n}{\sum_1^n \log x_i - n \log c}$ //

(c)

$L(\theta) = f(\underline{x};\theta) = c^n \theta^{nc} \left[\prod_{i=1}^n x_i\right]^{-(c+1)} I_{[x_{(1)} \geq \theta]}$ where $x_{(1)} = \min_i x_i$

Since $L(\theta)$ is increasing in $\theta$ for $\theta \leq x_{(1)}$ and has zero for $\theta > x_{(1)}$,

$\hat{\theta}_{MLE} = X_{(1)}$ //

(d)

$L(\theta) = f(\underline{x};\theta) = \theta^{\frac{n}{2}} \left[\prod_{i=1}^n x_i\right]^{\sqrt{\theta}-1}$

$\Rightarrow \ell(\theta) \overset{let}{=} \log L(\theta) = \frac{n}{2} \log \theta + (\sqrt{\theta}-1) \sum_1^n \log x_i$

$\Rightarrow \ell'(\theta) = \frac{n}{2\theta} + \frac{1}{2}\theta^{-\frac{1}{2}} \sum_1^n \log x_i \overset{set}{=} 0. \qquad \Rightarrow n + \theta^{\frac{1}{2}} \sum_1^n \log x_i = 0 \qquad \therefore \hat{\theta} = \left[\frac{n}{\sum_1^n \log x_i}\right]^2$

Since $\ell''(\theta) = -\frac{n}{2\theta^2} - \frac{1}{4}\theta^{-\frac{3}{2}} < 0$ for $\forall \theta > 0$, $\hat{\theta}_{MLE} = \left[\frac{n}{\sum_1^n \log x_i}\right]^2$ //

#2.1.11.

For $X \sim \Gamma(\alpha, \lambda)$,

$$EX = \frac{\alpha}{\lambda} \equiv \mu_1 \quad \cdots ①$$

$$EX^3 = \int_0^\infty x^3 \cdot \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+3)}{\lambda^{\alpha+3}} \underbrace{\int_0^\infty \frac{\lambda^{\alpha+3}}{\Gamma(\alpha+3)} x^{\alpha+3-1} e^{-\lambda x} dx}_{\overset{\shortparallel}{1}}$$

$$= \frac{(\alpha+2)(\alpha+1)\alpha\,\Gamma(\alpha)}{\Gamma(\alpha)} \cdot \frac{1}{\lambda^3} = \frac{\alpha(\alpha+1)(\alpha+2)}{\lambda^3} \equiv \mu_3 \quad \cdots ②$$

$$\Rightarrow \quad \mu_1 = \frac{\alpha}{\lambda}$$

$$\mu_3 = \frac{\alpha}{\lambda} \cdot \frac{\alpha+1}{\lambda} \cdot \frac{\alpha+2}{\lambda} = \mu_1 \left(\mu_1 + \frac{1}{\lambda}\right)\left(\mu_1 + \frac{2}{\lambda}\right)$$

$$\Rightarrow \quad (\mu_1^3 - \mu_3)\lambda^2 + 3\mu_1^2 \lambda + 2\mu_1 = 0$$

$$\therefore \quad \lambda = \frac{-3\mu_1^2 + \sqrt{9\mu_1^4 - 4(\mu_1^3 - \mu_3)(2\mu_1)}}{2(\mu_1^3 - \mu_3)} = \frac{-3\mu_1^2 + \sqrt{\mu_1^9 + 8\mu_1\mu_3}}{2(\mu_1^3 - \mu_3)}, \quad (\text{since } \lambda > 0)$$

$$\alpha = \frac{-3\mu_1^3 + \mu_1\sqrt{\mu_1^4 + 8\mu_1\mu_3}}{2(\mu_1^3 - \mu_3)}$$

Now, pluggng in $\bar{x}_n$ and $\frac{1}{n}\sum_1^n x_i^2$ for $\mu_1 (= EX)$ and $\mu_3 (= EX^3)$, respectnely,

$$\hat{\lambda} = \frac{-3\bar{x}_n^2 + \sqrt{\bar{x}_n^4 + 8\bar{x}_n\left(\frac{1}{n}\sum_1^n x_i^3\right)}}{2\left(\bar{x}_n^3 - \frac{1}{n}\sum_1^n x_i^3\right)},$$

$$\hat{\alpha} = \frac{-3\bar{x}_n^3 + \bar{x}_n\sqrt{\bar{x}_n^4 + 8\bar{x}_n\left(\frac{1}{n}\sum_1^n x_i^3\right)}}{2\left(\bar{x}_n^3 - \frac{1}{n}\sum_1^n x_i^3\right)} \quad //$$

# 2.2.14

$$L(\underset{\sim}{\theta}) = f(x|\underset{\sim}{\theta}) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

$$\Rightarrow \ell(\underset{\sim}{\theta}) = \log L(\underset{\sim}{\theta}) = -\frac{1}{2}\log 2\pi\sigma^2 - \frac{1}{2\sigma^2}(x-\mu)^2$$

$$\Rightarrow \begin{cases} \dfrac{\partial \ell(\underset{\sim}{\theta})}{\partial \mu} = \dfrac{1}{\sigma^2}(x-\mu) \overset{\text{set}}{=} 0 & \cdots ① \\[2ex] \dfrac{\partial \ell(\underset{\sim}{\theta})}{\partial \sigma^2} = -\dfrac{1}{2\sigma^2} + \dfrac{1}{2}(x-\mu)^2 \cdot \dfrac{1}{\sigma^4} \overset{\text{set}}{=} 0 & \cdots ② \end{cases}$$

From ① $\hat{\mu} = x$. but ② cannot be solved If $\mu = x$.

$\therefore$ $\nexists$ MLE of $\underset{\sim}{\theta}$ when $n=1$.

* Let $\mu = x$

$$\Rightarrow L(\underset{\sim}{\theta}) = \frac{1}{\sqrt{2\pi\sigma^2}} \longrightarrow \infty \quad \text{as} \quad \sigma^2 \to 0$$

I.e., no maxmum of the likelihood fct exists. //

⑥

(e)

$$L(\theta) = f(\underset{\sim}{x}|\theta) = \left[\prod_{i=1}^{n} x_i\right] \theta^{-2n} \cdot \exp\left[-\frac{1}{2\theta^2}\sum_{1}^{n} x_i^2\right]$$

$$\Rightarrow \ell(\theta) \overset{let}{=} \log L(\theta) = \sum_{1}^{n} \log x_i - 2n \log\theta - \frac{1}{2\theta^2}\sum_{1}^{n} x_i^2$$

$$\Rightarrow \ell'(\theta) = -\frac{2n}{\theta} + \frac{1}{2}\sum_{1}^{n} x_i^2 \cdot \frac{2\theta}{\theta^4} = -\frac{2n}{\theta} + \frac{\sum_{1}^{n} x_i^2}{\theta^3} \overset{set}{=} 0. \quad \therefore \hat{\theta} = \sqrt{\frac{\sum_{1}^{n} x_i^2}{2n}} \quad (\text{as } \theta > 0)$$

Since
$$\ell''(\theta) = \frac{2n}{\theta^2} + \sum_{1}^{n} x_i^2 \cdot \left(\frac{-3\theta^2}{\theta^6}\right) = \frac{2n}{\theta^2} - \frac{3\sum_{1}^{n} x_i^2}{\theta^4} \quad \text{and}$$

$$\ell''(\hat{\theta}) = \frac{2n}{\hat{\theta}^2}\left[1 - \frac{3\sum_{1}^{n} x_i^2}{2n\hat{\theta}^2}\right] = -\frac{4n}{\hat{\theta}^2} < 0, \qquad \hat{\theta}_{MLE} = \sqrt{\frac{\sum_{1}^{n} x_i^2}{2n}} \quad //$$

(f)

$$L(\theta) = f(\underset{\sim}{x}|\theta) = \theta^n c^n \left[\prod_{i=1}^{n} x_i\right]^{c-1} \exp\left[-\theta\sum_{1}^{n} x_i^c\right]$$

$$\Rightarrow \ell(\theta) = n\log\theta + n\log c + (c-1)\sum_{1}^{n}\log x_i - \theta\sum_{1}^{n} x_i^c$$

$$\Rightarrow \ell'(\theta) = \frac{n}{\theta} - \sum_{1}^{n} x_i^c \overset{set}{=} 0, \qquad \therefore \hat{\theta} = \frac{n}{\sum_{1}^{n} x_i^c}$$

Since $\ell''(\theta) = -\frac{n}{\theta^2} < 0$ for $\forall \theta > 0$, $\quad \hat{\theta}_{MLE} = \frac{n}{\sum_{1}^{n} x_i^c} \quad //$

#2.1.14

(a)

$$EX_i = \mu + Ee_i = \mu + E[\beta e_{i-1} + \varepsilon_i] = \mu + \beta \cdot Ee_{i-1} + E\varepsilon_i$$

Since $E\varepsilon_i = 0$, $\beta \cdot Ee_{i-1} = \beta[\beta Ee_{i-2}] = \cdots = \beta^i Ee_0 = 0$, $EX_i = \mu$, $\forall i = 1, \cdots, n$

$$\Rightarrow \quad \mu \overset{set}{=} \bar{x}_n. \qquad \therefore \hat{\mu}_{MME} = \bar{x}_n. \quad //$$

(b) i) $EU_i = \dfrac{1}{\sqrt{\sum_{j=0}^{i-1} b^{2j}}} [EX_i - \mu_0] = 0 \qquad \therefore EU_i^2 = \mathrm{Var}\,U_i.$

ii) $x_i - \mu_0 = e_i = \beta e_{i-1} + \varepsilon_i = \beta[\beta e_{i-2} + \varepsilon_{i-1}] + \varepsilon_i = \cdots$

$$= \beta^i e_0 + \beta^{i-1}\varepsilon_1 + \beta^{i-2}\varepsilon_2 + \cdots + \beta\varepsilon_{i-1} + \varepsilon_i$$

$$\underset{\substack{\because\, \beta = b, \\ e_0 = 0}}{=} b^{i-1}\varepsilon_1 + b^{i-2}\varepsilon_2 + \cdots + b\varepsilon_{i-1} + \varepsilon_i$$

$$\Rightarrow \mathrm{Var}(X_i - \mu_0) = \sigma^2[1 + b^2 + \cdots + b^{2i-2}] = \left[\sum_{j=0}^{i-1} b^{2j}\right]\sigma^2.$$

$$\Rightarrow E U_i^2 \overset{i)}{=} \mathrm{Var}\,U_i = \frac{1}{\sum_{j=0}^{i-1} b^{2j}} \mathrm{Var}(X_i - \mu_0) = \sigma^2.$$

$$\therefore \hat{\sigma}^2_{MME} = \frac{1}{n}\sum_1^n U_i^2 \quad //$$

(c)

$$\text{Since} \quad \beta = \frac{\mathrm{Cov}(X_i - \mu, X_{i-1} - \mu)}{\mathrm{Var}(X_{i-1} - \mu)} \left( = \frac{\sum_{j=0}^{i-2} \beta^{2j+1} \cdot \cancel{\sigma^2}}{\sum_{j=0}^{i-2} \beta^{2j} \cdot \cancel{\sigma^2}} \right),$$

$$\hat{\beta}_{MME} = \frac{\sum_{i=2}^{n} (x_i - \bar{x})(x_{i-1} - \bar{x}) / n-2}{\sum_{i=2}^{n} (x_{i-1} - \bar{x})^2 / n-1}. \quad //$$

#2.2.15

$L(\theta) = f(x|\theta) = g(T(x)|\theta) \cdot h(x)$

$\Rightarrow l(\theta) = \log g(T(x)|\theta) + \log h(x)$

MLE of $\theta$, say $\hat{\theta}$, maximizes $l(\theta)$. and we can see that $\hat{\theta}$ depends on $X$ only through $T(X)$. //

#2.2.19 (a)

$$L(\underset{\sim}{\theta}) = \prod_{i=1}^{n} f(x_i|\theta_i) = (2\pi)^{-\frac{n}{2}} \exp\left[-\frac{1}{2}\sum_{i=1}^{n}(x_i-\theta_i)^2\right]$$

$$\Rightarrow l(\underset{\sim}{\theta}) \overset{let}{=} \log L(\underset{\sim}{\theta}) = -\frac{n}{2}\log 2\pi - \frac{1}{2}\sum_{i=1}^{n}(x_i-\theta_i)^2$$

$$\Rightarrow \frac{\partial l(\underset{\sim}{\theta})}{\partial \theta_i} = x_i - \theta_i \overset{set}{=} 0 \qquad \therefore \hat{\theta}_i = x_i \quad \text{for each } i.$$

(Note that $\frac{\partial l(\underset{\sim}{\theta})}{\partial\underset{\sim}{\theta}\partial\underset{\sim}{\theta}} = -I_{n\times n} \Rightarrow \left|\frac{\partial l(\underset{\sim}{\theta})}{\partial\underset{\sim}{\theta}\partial\underset{\sim}{\theta}'}\right| = 1 > 0$, $\therefore \hat{\theta}_i = x_i$ is the MLE )

(b)

i) For $x_1 \le x_2$, $\hat{\theta}_1 = x_1$ and $\hat{\theta}_2 = x_2$ from (a).

ii) For $x_1 > x_2$,

$$l(\underset{\sim}{\theta}) = -\log 2\pi - \frac{1}{2}\sum_{1}^{2}(x_i-\theta_i)^2 = -\log 2\pi - \frac{1}{2}\left[(x_1^2+x_2^2) - 2(x_1\theta_1+x_2\theta_2) + \theta_1^2+\theta_2^2\right]$$

Since $(x_1\theta_1 + x_2\theta_2) \overset{\text{Cauchy-Schwarz ineq}}{\le} \sqrt{(x_1^2+x_2^2)(\theta_1^2+\theta_2^2)} \underset{\because\, \theta_1 \le \theta_2}{\le} \sqrt{(x_1^2+x_2^2)\cdot 2\theta_2^2}$,

$l(\underset{\sim}{\theta})$ is maximized when $\theta_1 = \theta_2$.

$$\Rightarrow l(\theta_1=\theta_2) = -\log 2\pi - \frac{1}{2}\sum_{1}^{2}(x_i-\theta_2)^2$$

$$\Rightarrow l'(\theta_2) = \sum_{1}^{2}(x_i-\theta_2) = \sum_{1}^{2} x_i - 2\theta_2 \overset{set}{=} 0 \qquad \therefore \hat{\theta}_1 = \hat{\theta}_2 = \frac{\sum x_i}{2}$$

Since $l''(\theta_2) = -2 < 0$ for $\forall \theta_2$, $\hat{\theta}_1 = \hat{\theta}_2 = \frac{\sum_{1}^{2} x_i}{2}$ is MLE //.

#2.2.16 (a)

Since $h$ is one-to-one fct from $\Theta$ onto $h(\Theta)$, then $h$ is invertible., ($\exists\ h^{-1}$).

Thus, $f(x,\theta)$ can also be written as a form of $g(x,\eta) \triangleq f(x, h^{-1}(\eta))$.

Since $\hat{\theta}$ is the MLE, for any $\theta \in \Theta$, $f(x,\hat{\theta}) \geq f(x,\theta)$.

$\Rightarrow$ $f(x, h^{-1}(\hat{\eta})) \geq f(x, h^{-1}(\eta))$ , $\forall \eta \in h(\Theta) = \{h(\theta) : \theta \in \Theta\}$. where $\hat{\eta} = h(\hat{\theta})$.

$\Rightarrow$ $g(x,\hat{\eta}) \geq g(x,\eta)$ , $\forall \eta \in h(\Theta)$.

$\therefore$ $\hat{\eta}$ is the MLE. //

(b)

Let $\Theta(\omega) = \{\theta \in \Theta : g(\theta) = \omega\} \subset \Theta$. and $\Theta(\hat{\omega})$ satisfy $g(\Theta(\hat{\omega})) = g(\hat{\theta}) = \hat{\omega}$.

Let $\hat{\omega}_{MLE} = \arg\sup_{\omega \in \Omega} \sup [L_X(\theta),\ \theta \in \Theta(\omega)]$.

Since $\sup_{\omega \in \Omega} \sup [L_X(\theta),\ \theta \in \Theta(\omega)] = L_X(\hat{\theta})$ as $\hat{\theta}$ is a MLE of $\theta$,

$\hat{\omega}_{MLE} = g(\Theta(\hat{\omega})) = g(\hat{\theta}) = \hat{\omega}$.

$\therefore$ $\hat{\omega}_{MLE} = \hat{\omega} = g(\hat{\theta})$ //

10

# 2.2.21.

① WLOG, let $\hat{\mu} = x_1$.

i) $f_1(x_1, \theta) = \frac{9}{10} \cdot \frac{1}{\sqrt{2\pi}\sigma} + \frac{1}{10} \cdot \frac{1}{\sqrt{2\pi}} \longrightarrow \infty$ as $\sigma^2 \longrightarrow 0$

$$f_i(x_i, \theta) = \frac{9}{10} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x_i - x_1)^2}{2\sigma^2}} + \frac{1}{10} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_i - x_1)^2}{2}}$$

$$\longrightarrow \frac{1}{10\sqrt{2\pi}} e^{-\frac{(x_i - x_1)^2}{2}} \quad \text{as} \quad \sigma \longrightarrow 0 \qquad \text{for } i = 2, 3, \ldots, n$$

$\therefore$ $L(\theta) = \prod_{i=1}^{n} f_i(x_i, \theta) \longrightarrow \infty$ as $\sigma \longrightarrow 0$.

The likelihood fct is unbounded, which means that the MLE of $(\mu, \sigma^2)$ does not exist.

② From i), we have $\sup_{\sigma} f(\underline{x}, \hat{\mu}, \sigma) = \sup_{\sigma, \mu} f(\underline{x}, \mu, \sigma) = \infty$ for $\hat{\mu} = x_1$ (WLOG)

If possible, suppose that $\hat{\mu} \neq x_i$, $\forall i$, $1 \le i \le n$.

Then $\frac{1}{\sigma} \exp\left(-\frac{M}{\sigma^2}\right) \longrightarrow 0$ as $\sigma \longrightarrow 0$ where $M = \max_i (x_i - \mu)^2$.

$\Rightarrow$ $\sup_{\sigma} f(\underline{x}, \hat{\mu}, \sigma) < \infty$ $\ast$

$\therefore$ $\sup_{\sigma} f(\underline{x}, \hat{\mu}, \sigma) = \sup_{\sigma, \mu} f(\underline{x}, \mu, \sigma)$ iff $\hat{\mu} = x_1$ (WLOG).