STAT 543

HW 6 solution

Spring 2016

#1.

sol) ① i) $E[\delta^*(X)] = E[1+\bar{X}^2 - \frac{1}{n}] = 1 + E\bar{X}^2 - \frac{1}{n} = 1 + V(\bar{X}) + (E\bar{X})^2 - \frac{1}{n} = 1 + \frac{1}{n} + \theta^2 - \frac{1}{n} = 1+\theta^2$

$\therefore \delta^*(X)$ is unbiased for $\gamma(\theta)$.

ii)

$$f(x;\theta) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}(x-\theta)^2\} = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}x^2 + x\theta - \frac{1}{2}\theta^2\}$$

which belongs to exponential family and $\Theta = \mathbb{R}$ contains an open interval.

$\Rightarrow \sum_1^n X_i$ is complete and sufficient for θ.

$\delta^*(X)$ is unbiased and a fct of $\sum_1^n X_i$, c.s.s.

$\therefore$ By Lehmann - Scheffe's thm, $\delta^*(X)$ is the UMVUE of $\gamma(\theta)$.

② $E[(\delta^*(X) - \gamma(\theta))^2 - (\delta^{**}(X) - \gamma(\theta))^2]$

$= E[(1+\bar{X}^2 - \frac{1}{n} - \gamma(\theta))^2 - (1+\bar{X}^2 - \frac{1}{n} - \gamma(\theta))^2 I_{[\bar{X}^2 > \frac{1}{n}]} - (1-\gamma(\theta))^2 I_{[\bar{X}^2 \le \frac{1}{n}]}]$

$= E[\{(1+\bar{X}^2 - \frac{1}{n} - \gamma(\theta))^2 - (1-\gamma(\theta))^2\} I_{[\bar{X}^2 \le \frac{1}{n}]}]$

$= E[(2+\bar{X}^2 - \frac{1}{n} - 2\gamma(\theta))(\bar{X}^2 - \frac{1}{n}) I_{[\bar{X}^2 \le \frac{1}{n}]}] > 0, \quad \forall\theta$.

$\Rightarrow E[(\delta^*(X) - \gamma(\theta))^2] > E[(\delta^{**}(X) - \gamma(\theta))^2], \quad \forall\theta$

$\therefore MSE_\theta\, \delta^*(X) > MSE_\theta\, \delta^{**}(X), \quad \forall\theta$.

③ $E_\theta[\delta^{**}(X)] = E_\theta[\delta^*(X) I_{[\bar{X}^2 > \frac{1}{n}]} + (\delta^*(X) - \bar{X}^2 + \frac{1}{n}) I_{[\bar{X}^2 \le \frac{1}{n}]}]$

$= E_\theta \delta^*(X) + E_\theta[(\frac{1}{n} - \bar{X}^2) I_{[\bar{X}^2 \le \frac{1}{n}]}]$

$= \gamma(\theta) + E_\theta[(\frac{1}{n} - \bar{X}^2) I_{[\bar{X}^2 \le \frac{1}{n}]}] > \gamma(\theta)$

$\therefore$ $\delta^{**}(X)$ is not unbiased. //

#2.

① Let $Y_1 = -\ln X_1$. $\Rightarrow$ $x_1 = e^{-y_1}$, $\left|\frac{dx_1}{dy_1}\right| = e^{-y_1}$ and $0 < y_1 < \infty$

$g(y_1|\theta) = f(e^{-y_1}|\theta) \cdot e^{-y_1} = \theta e^{-y_1(\theta-1)} \cdot e^{-y_1} = \theta e^{-\theta y_1}$, $0 < y_1 < \infty$.

$\therefore$ $Y_1 \sim Exp(\frac{1}{\theta})$ and $EY_1 = \frac{1}{\theta}$

② $f(\underset{\sim}{x}|\theta) = \prod_{i=1}^n \theta x_i^{\theta-1} = \theta^n \left[\prod_{i=1}^n x_i\right]^{\theta-1} = \exp\left[n\ln\theta + (\theta-1)\sum_1^n \ln x_i\right]$

which belongs to exponenential family and natural parameter space contains an open interval, $\sum_1^n \ln X_i$ is C.S.S. for θ.

Since $E_\theta[-\frac{1}{n}\sum_1^n \ln X_i] = E_\theta[-\ln X_1] = \frac{1}{\theta}$,

by Lemann - Scheffe thm, $-\frac{1}{n}\sum_1^n \ln X_i$ is UMVUE

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#7. (a)

① $f(x_1, x_2 | \theta) = \frac{1}{\theta^2} I_{[1 \le x_1 \le \theta]} I_{[1 \le x_2 \le \theta]} = \frac{1}{\theta^2} I_{[1 \le \min(x_1, x_2)]} I_{[\max(x_1, x_2) \le \theta]}$

$= g(y(x); \theta) \cdot h(x)$ where $g(y; \theta) = \frac{1}{\theta^2} I_{[y \le \theta]}$, $h(x) = I_{[1 \le \min(x_1, x_2)]}$ and

$Y(x) = \max(x_1, x_2)$

By factorization thm, $Y(\underset{\sim}{x}) = \max(x_1, x_2)$ is sufficient for θ.

② claim: $E_\theta[h(Y)] = 0$ for $\forall_\theta$ $\Rightarrow$ $P_\theta(h(Y) = 0) = 1$, $\forall_\theta$.

i) For $Y = \max(x_1, x_2)$,

$P_\theta(Y=y) = P_\theta(Y \le y) - P_\theta(Y \le y-1) = P_\theta(X_1 \le y) P_\theta(X_2 \le y) - P_\theta(X_1 \le y-1) P_\theta(X_2 \le y-1)$

$= [\frac{y}{\theta}]^2 - [\frac{y-1}{\theta}]^2 = \frac{2y-1}{\theta^2}$, $y = 1, 2, \cdots, \theta$.

ii) Suppose that $E_\theta[h(Y)] = 0$ for $\forall_\theta$.

$\Rightarrow$ $0 = E_\theta[h(Y)] = \sum_{y=1}^{\theta} h(y) \cdot \frac{2y-1}{\theta^2}$ for $\forall_\theta$.

Let $\theta = 1$. $\Rightarrow$ $h(1) \cdot \frac{1}{\theta^2} = 0$ for all θ. This implies $h(1) = 0$. (a)

Let $\theta = 2$ $\Rightarrow$ $h(1) \cdot \frac{1}{\theta^2} + h(2) \cdot \frac{3}{\theta^2} = 0$ for all θ. $\Rightarrow$ $h(2) = 0$ as $h(1) = 0$ by (a)

⋮

From the similar arguments, we can get $h(y) = 0$ for any y for $\forall_\theta$.

$\Rightarrow$ $P_\theta(h(Y) = 0) = 1$, $\forall_\theta$.

$\therefore$ $Y = \max(X_1, X_2)$ is a C.S.S. of θ by ① and ② //

(b)

Method 1) Since $\delta(X)$ is a fct of Y which is a C.S.S. of θ,

by Lehmann – Scheffe Thm, It is enough to show that $E_\theta \delta(X) = \theta$.

$$E_\theta \delta(X) = E_\theta \left[\frac{Y^3-(Y-1)^3}{Y^2-(Y-1)^2} \right] = \sum_{y=1}^{\theta} \frac{y^3-(y-1)^3}{y^2-(y-1)^2} \cdot P_\theta(Y=y) = \sum_{y=1}^{\theta} \frac{3y^2-3y+1}{2y-1} \cdot \frac{2y-1}{\theta^2}$$

$$= \frac{1}{\theta^2}\left[3\sum_{y=1}^{\theta} y^2 - 3\sum_{y=1}^{\theta} y + \sum_{y=1}^{\theta} 1 \right] = \frac{1}{\theta^2}\left[3\cdot\frac{\theta(\theta+1)(2\theta+1)}{6} - 3\cdot\frac{\theta(\theta+1)}{2} + \theta \right] = \theta.$$

Method 2) Note that $2X_1 - 1$ is unbiased for θ.

$$\because \quad E_\theta X_1 = \sum_{x=1}^{\theta} x\cdot\frac{1}{\theta} = \frac{1}{\theta}\cdot\frac{\theta(\theta+1)}{2} = \frac{\theta+1}{2} \quad \Rightarrow \quad E_\theta[2X_1-1] = 2\cdot E_\theta X_1 - 1 = \theta.$$

$$\Rightarrow E_\theta[2X_1-1 \mid Y=y] = 2\cdot E_\theta[X_1\mid Y=y] - 1 = 2\cdot\left[y\cdot P_\theta(X_1=y\mid Y=y) + \sum_{k=1}^{y-1} k\cdot P_\theta(X_1=k\mid Y=y) \right] - 1$$

$$= 2\cdot\left[y\cdot\frac{P_\theta(X_1=y, X_2\le y)}{P_\theta(Y=y)} + \sum_{k=1}^{y-1} k\cdot\frac{P(X_1=k, X_2=y)}{P_\theta(Y=y)} \right] - 1$$

$$= 2\cdot\left[y\,\frac{\frac{1}{\theta}\cdot\frac{y}{\theta}}{\frac{2y-1}{\theta^2}} + \sum_{k=1}^{y-1} k\cdot\frac{\frac{1}{\theta}\cdot\frac{1}{\theta}}{\frac{2y-1}{\theta^2}} \right] - 1$$

$$= 2\left[\frac{y^2}{2y-1} + \frac{1}{2y-1}\cdot\frac{(y-1)y}{2} \right] - 1 = \frac{3y^2-3y+1}{2y-1} = \delta(x)$$

By Rao – Blackwell thm, $\delta(X) = \dfrac{Y^3-(Y-1)^3}{Y^2-(Y-1)^2}$ is a UMVUE of θ. //

(c)

$$E[I(X_1=1)\mid Y=y] = P(X_1=1\mid Y=y) = \frac{P_\theta(X_1=1, X_2=y)}{P_\theta(Y=y)} = \frac{\frac{1}{\theta}\cdot\frac{1}{\theta}}{\frac{2y-1}{\theta^2}} = \frac{1}{2y-1}$$

* since $y\in\{1,2,\cdots,\theta\}$ and $X_1=1$,
X_2 should be y

Since $I(X_1=1)$ is an unbiased estimator of $\frac{1}{\theta}$ and Y is a c.s.s. for θ,

$E[I(X_1=1) \mid Y] = \frac{1}{2Y-1}$ is a UMVUE of $\frac{1}{\theta}$ by Rao-Blackwell thm. //

#4. (a)

i) $f(x;\beta) = \exp[-\frac{x}{\beta} - \ln\beta]$

which belongs to an exponential family with the natural parameter space

containing an open interval, $\sum_1^n X_i$ is complete and sufficient for β.

ii) Since $X_i \overset{iid}{\sim}$ Gamma$(1, \beta)$, $\sum_1^n X_i \sim$ Gamma(n, β).

For $\frac{X_1}{\sum_1^n X_i}$, $\frac{X_1}{\sum_1^n X_i} = \frac{X_1}{X_1 + \sum_{i=2}^n X_i} \sim$ beta$(1, n-1)$. ..... (*)

Since the dist. of $\frac{X_1}{\sum_1^n X_i}$ does not depend on β,

$\frac{X_1}{\sum_1^n X_i}$ is an ancillary stat. of β.

By Basu thm, $\sum_1^n X_i$ and $\frac{X_1}{\sum_1^n X_i}$ are independent. //

(*) : claim: For $W \sim$ Gamma$(1, \beta)$ and $Z \sim$ Gamma$(n-1, \beta)$,

$Y \equiv \frac{W}{W+Z} \sim$ beta$(1, n-1)$.

Consider $X = W$ and $Y = \frac{W}{W+Z}$. Then, we get $W = X$, $Z = \frac{X}{Y} - X$ and

$$J = \begin{vmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{1}{y}-1 & -\frac{x}{y^2} \end{vmatrix} = -\frac{x}{y^2} \quad \therefore |J| = \frac{x}{y^2}.$$

$$\Rightarrow g_{X,Y}(x,y;\beta) = f_{W,Z}\left(x, \frac{x}{y}-x\right)\cdot|J| = \frac{1}{\beta}e^{-\frac{x}{\beta}} \times \frac{1}{\beta^{n-1}\cdot\Gamma(n-1)}\left(\frac{x}{y}-x\right)^{n-2} e^{-\frac{1}{\beta}\left(\frac{x}{y}-x\right)} \cdot \frac{x}{y^2}$$

$$= \frac{1}{\Gamma(n-1)\,\beta^n}\cdot\frac{1}{y^n}\,x^{n-1}(1-y)^{n-2}e^{-\frac{x}{\beta y}}, \quad 0<x<\infty, \quad 0<y<1.$$

$$\Rightarrow g_Y(y) = \int_0^\infty \frac{1}{\Gamma(n-1)\,\beta^n}\cdot\frac{1}{y^n}\,x^{n-1}(1-y)^{n-2}\cdot e^{-\frac{x}{\beta y}}\,dx$$

$$= \frac{1}{\Gamma(n-1)\,\beta^n}\cdot\frac{1}{y^n}(1-y)^{n-2}\int_0^\infty x^{n-1}e^{-\frac{x}{\beta y}}\,dx$$

$$= \frac{1}{\Gamma(n-1)\,\beta^n}\,\frac{(1-y)^{n-2}}{y^n}\cdot\Gamma(n)(\beta y)^n = (n-1)(1-y)^{n-2}, \quad 0<y<1 \quad \therefore Y \sim \text{beta}(1, n-1)$$

(b)

$$\delta(T) \equiv E\left[I(X_1>t) \,\middle|\, \sum_1^n X_i = T\right] = P\left(X_1>t \,\middle|\, \sum_1^n X_i = T\right)$$

$$= P\left(\frac{X_1}{\sum_1^n X_i} > \frac{t}{\sum_1^n X_i} \,\middle|\, \sum_1^n X_i = T\right)$$

$$= P\left(\frac{X_1}{\sum_1^n X_i} > \frac{t}{T}\right) \quad \text{as} \quad \sum_1^n X_i \perp \frac{X_1}{\sum_1^n X_i} \quad \text{by (a).}$$

$$= \int_{\frac{t}{T}}^1 (n-1)(1-y)^{n-2}\,dy = \left(1-\frac{t}{T}\right)^{n-1}$$

$\therefore$ $\delta(T) = \left(1-\frac{t}{T}\right)^{n-1}$ is a UMVUE of $\gamma(\beta) = e^{-\frac{t}{\beta}}$

↳ Since $I_{[X_1>t]}$ is an unbiased est. of $\gamma(\beta)$,

$\delta(T)$, which is a fct of a c.s.s. of β, is also unbiased for $\gamma(\beta)$. Then, by Lehmann – Scheffe thm, $\delta(T)$ is a UMVUE of $\gamma(\beta)$.

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3.4.12

(a)

i) $L(\theta) = f(\underset{\sim}{x}|\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n \left[\prod_{i=1}^{n} x_i \right]^{\theta-1}$

$\Rightarrow \ell(\theta) \equiv \ln L(\theta) = n \ln\theta + (\theta-1) \sum_1^n \ln x_i$

$\Rightarrow \ell'(\theta) = \frac{n}{\theta} + \sum_1^n \ln x_i \overset{\text{set}}{=} 0 \qquad \hat{\theta} = -\frac{n}{\sum_1^n \ln x_i}$

As $\ell''(\theta) = -\frac{n}{\theta^2} < 0$ for $\forall \theta$, $\hat{\theta}$ is MLE of θ.

By the invariance property of MLE, $\widehat{\frac{1}{\theta}} = -\frac{1}{n} \sum_1^n \ln x_i$ is MLE of $\frac{1}{\theta}$.

ii) Since the beta dist.,

$$f(x|\theta) = \exp[\ln\theta + (\theta-1)\ln x],$$

belongs to the exponential family,

$E[\ln X] = -\frac{1}{\theta}$ and

$E[-\frac{1}{n} \sum_1^n \ln X_i] = E[-\ln X] = \frac{1}{\theta}$.

$\therefore$ The MLE of $\frac{1}{\theta}$, $\widehat{\frac{1}{\theta}}$, is unbiased for $\frac{1}{\theta}$ and achieves the Information inequality lower bound. //

(b)

$$E_\theta \bar{X} = E_\theta X_1 = \frac{\theta}{\theta+1}$$

$$\Rightarrow Var_\theta \bar{X} = \frac{1}{n} \cdot \frac{\theta}{(\theta+1)^2(\theta+2)} \quad \text{but}$$

$$\frac{\left[\frac{d}{d\theta}\frac{\theta}{\theta+1}\right]^2}{nI(\theta)} = \frac{\left[\frac{1}{(\theta+1)^2}\right]^2}{n \cdot \frac{1}{\theta^2}} = \frac{\theta^2}{n(\theta+1)^4} . \quad \text{i.e., } Var_\theta \bar{X} > \frac{\theta^2}{n(\theta+1)^4}$$

$\therefore$ $\bar{X}$ does not achieve the information inequality lower bound. //

#7.4.22.

pf) i)

$$p(x;\theta) = \frac{1}{\theta} I_{[0<x<\theta]} \quad \Rightarrow \quad \log p(x;\theta) = -\log\theta \cdot I_{[0<x<\theta]}$$

$$\Rightarrow \frac{\partial \log p(x;\theta)}{\partial\theta} = -\frac{1}{\theta} \quad \text{for} \quad \theta > x.$$

$$\Rightarrow E\left[\frac{\partial \log p(x;\theta)}{\partial\theta}\right] = -\frac{1}{\theta} \neq 0 . \quad \text{for} \quad \theta > x.$$

ii) $$Var\left[\frac{\partial \log p(x;\theta)}{\partial\theta}\right] = Var\left[-\frac{1}{\theta}\right] = 0$$ $\therefore$ The information inequality lower bound is infinite.

(Note : we can't use information for $I(\theta) = E\left[\left(\frac{\partial \log p(x;\theta)}{\partial\theta}\right)^2\right]$ anymore)

iii) $$E_\theta(2X) = 2 \cdot E_\theta X = 2 \cdot \int_0^\theta x \cdot \frac{1}{\theta} dx = \theta$$

$$Var_\theta(2X) = 4 \cdot Var_\theta X = 4 \cdot \left[\int_0^\theta x^2 \cdot \frac{1}{\theta} dx - \left(\frac{\theta}{2}\right)^2\right] = \frac{\theta^2}{3} < \infty .$$

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#3.4.5

(a)

i) $L(\sigma^2) = f(\underset{\sim}{x};\sigma^2) = \prod_{i=1}^{n} (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left[-\frac{1}{2\sigma^2}(x_i-\mu)^2\right] = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left[-\frac{1}{2\sigma^2}\sum_1^n (x_i-\mu)^2\right]$

$\Rightarrow \ell(\sigma^2) = -\frac{n}{2}\ln 2\pi\sigma^2 - \frac{1}{2\sigma^2}\sum_1^n (x_i-\mu)^2$

$\Rightarrow \frac{\partial \ell}{\partial \sigma^2} = -\frac{n}{2\sigma^2} - \frac{1}{2}\sum_1^n (x_i-\mu)^2 \cdot \left(-\frac{1}{\sigma^4}\right) = -\frac{n}{2\sigma^2} + \frac{\sum_1^n (x_i-\mu)^2}{2\sigma^4}$

$\Rightarrow \frac{\partial^2 \ell}{\partial (\sigma^2)^2} = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}\sum_1^n (x_i-\mu)^2$

Since $-E\left[\frac{n}{2\sigma^4} - \frac{1}{\sigma^6}\sum_1^n (x_i-\mu)^2\right] = \frac{n}{2\sigma^4}$, the information inequality lower bound for unbiased estimators of σ^2 is $\frac{2\sigma^4}{n}$.

ii) Note that $E[\hat{\sigma}_0^2] = E\left[\frac{1}{n}\sum_{i=1}^n (x_i-\mu_0)^2\right] = \sigma^2$

and since $\frac{n\hat{\sigma}_0^2}{\sigma^2} = \sum_1^n \left(\frac{x_i-\mu_0}{\sigma}\right)^2 = \sum_1^n Z_i^2 \sim \chi_n^2$ since $Z_i \overset{iid}{\sim} N(0,1)$,

$Var\left(\frac{n\hat{\sigma}_0^2}{\sigma^2}\right) = 2n \Rightarrow Var(\hat{\sigma}_0^2) = \frac{\sigma^4}{n^2}\cdot 2n = \frac{2\sigma^4}{n}$.

$\therefore \hat{\sigma}_0^2$ is UMVUE of σ^2.

(b)

Let $\delta(\underset{\sim}{X}) = \frac{1}{n+1} \sum_{i=1}^{n} (X_i - \overline{X})^2$.

$\Rightarrow \; E[\delta(\underset{\sim}{X})] = E[\frac{1}{n+1} \sum_{i=1}^{n} (X_i - \mu_0 + \mu_0 - \overline{X})^2] = \frac{1}{n+1} E[\sum_{i=1}^{n} (X_i - \mu_0)^2 + n \cdot (\overline{X} - \mu_0)^2]$

$= \frac{1}{n+1} [n\sigma^2 + n \cdot \frac{\sigma^2}{n}] = \sigma^2.$

Since $\frac{(n+1)\,\delta(\underset{\sim}{X})}{\sigma^2} = \frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{\sigma^2} \sim \chi^2_{n-1}$,

$Var[\frac{(n+1)}{\sigma^2} \delta(\underset{\sim}{X})] = 2(n-1) \quad \Rightarrow \quad Var(\delta(\underset{\sim}{X})) = \frac{2(n-1)\sigma^4}{(n+1)^2} < \frac{2\sigma^4}{n} = Var(\hat{\sigma}_0^2)$

Since $\delta(\underset{\sim}{X})$ and $\hat{\sigma}_0^2$ are unbiased for σ^2, $MSE(\delta(\underset{\sim}{X})) < MSE(\hat{\sigma}_0^2)$.

$\therefore$ $\hat{\sigma}_0^2$ is inadmissible under squared error loss.

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(c)

$Bias(\hat{\sigma}_0^2) = E[\hat{\sigma}_0^2] - \sigma^2 = E[\frac{1}{n} \sum_{i=1}^{n} (X_i - \mu_0)^2] - \sigma^2 = E[\frac{1}{n} \sum_{i=1}^{n} (X_i - \mu + \mu - \mu_0)^2] - \sigma^2$

$= \underbrace{\frac{1}{n} \sum_{i=1}^{n} E[(X_i - \mu)^2]}_{= \sigma^2} + (\mu - \mu_0)^2 - \sigma^2 = (\mu - \mu_0)^2.$

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#7.4.20.

(a)

pf) $\Leftarrow$: Suppose that $P[\delta(x)=\theta]=1$. for any θ.

i) for any fixed θ, $E_\theta \delta(x) = \theta \cdot P[\delta(x)=\theta] = \theta$.

ii) for given x, $E[\theta|x] = \theta \cdot P[\theta=\delta(x)] = \delta(x)$ //

$\Rightarrow$: Suppose that $E_\theta \delta(x)=\theta$. and $E[\theta|x]=\delta(x)$.

$$E[\delta(x)-\theta]^2 = E[\delta(x)^2 - 2\delta(x)\cdot\theta + \theta^2] = E[\delta(x)^2 - \delta(x)\cdot\theta] + E[\theta^2 - \delta(x)\cdot\theta]$$

$\uparrow$

wrt (x,θ)

$$= E[E[\delta(x)^2 - \delta(x)\cdot\theta \mid x]] + E[E[\theta^2 - \delta(x)\cdot\theta \mid \theta]]$$

$$= E[\delta(x)^2 - \delta(x)\cdot \underset{\delta(x)}{\underset{\shortparallel}{E[\theta|x]}}] + E[\theta^2 - \theta\cdot \underset{\theta}{\underset{\shortparallel}{E[\delta(x)|\theta]}}] = 0$$

$\Rightarrow$ $P(\delta(x)-\theta=0)=1$.

(b)

pf) If possible, suppose that X is a Bayes estimator for some prior π.

$\Rightarrow$ $E[\theta|X]=X$ and $E_\theta X=\theta$.

Then, from (a), $P[X=\theta]=1$ $\times$ since $X\sim N(\theta,\sigma_0^2)$.

(c)

We know that $\theta|x \sim beta(x, n-x) \Rightarrow E[\theta|x] = \frac{x}{n}$.

For $\pi(\theta) \propto \theta^{-1}(1-\theta)^{-1}$,

$$\pi(\theta|x) \propto p(x|\theta)\pi(\theta) \propto \theta^{x}(1-\theta)^{n-x} \cdot \theta^{-1}(1-\theta)^{-1} = \theta^{x-1}(1-\theta)^{n-x-1}$$

implies $\theta|x \sim beta(x, n-x)$.

In the case of $\pi(\theta) \propto \theta^{-1}(1-\theta)^{-1}$, $\frac{x}{n}$ is a Bayes est. of θ.

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