Stat 543 Assignment 6 (Due , 2016)
Rao-Blackwell, Lehmann-Scheffé, Cramér-Rao Inequality

1. Suppose that $X_1, X_2, \ldots, X_n$ are iid $\mathrm{N}(\theta, 1)$ and that $\gamma(\theta) = \mathrm{E}_\theta X_1^2 = \theta^2 + 1$ is of interest. In class Vardeman Rao-Blackwellized a method of moments estimator of $\gamma(\theta)$ and obtained the estimator $\delta^*(X) = 1 + \overline{X}^2 - \frac{1}{n}$. Argue that this (silly) estimator of $\gamma(\theta)$ is the UMVUE. Argue carefully that the estimator

$$\delta^{**}(X) = \begin{cases} \delta^*(X) & \text{if } \overline{X}^2 > \frac{1}{n} \\ 1 & \text{if } \overline{X}^2 \leq \frac{1}{n} \end{cases}$$

is strictly better than $\delta^*(X)$ (that is, $\mathrm{MSE}_\theta(\delta^{**}(X)) < \mathrm{MSE}_\theta(\delta^*(X))\ \forall\theta$.) (Hint: What can you say about the random variable $(\delta^*(X) - \gamma(\theta))^2 - (\delta^{**}(X) - \gamma(\theta))^2$?) Is $\delta^{**}(X)$ unbiased?

2. (Gartwaithe, Jolliffe, and Jones) Suppose that $X_1, X_2, \ldots, X_n$ are iid with marginal pdf
$$f(x|\theta) = I[0 \leq x \leq 1]\, \theta x^{\theta-1}$$
for $\theta > 0$. $-\ln X_1$ is an unbiased estimator of θ^{-1}. Find a better one (one with smaller variance). Is your estimator UMVU? Explain.

3. As an example of an ad hoc (non-exponential-family) application of the Lehmann-Scheffé argument, consider the following. Suppose that X_1, X_2 are iid with marginal pmf on the positive integers
$$f(x|\theta) = \frac{1}{\theta} I[1 \leq x \leq \theta]$$
for θ a positive integer.

   (a) Show that the statistic $Y = \max(X_1, X_2)$ is both sufficient and complete. (Look on the handouts for Bahadur's Theorem and the Lehmann-Scheffé Theorem for a definition of completeness.) (Think about this problem in geometric terms (on the grid of integer pairs in the (x_1, x_2)-plane) in order to work out the distribution of Y.)

   (b) Then argue that
$$\delta(X) = \frac{Y^3 - (Y-1)^3}{Y^2 - (Y-1)^2}$$
is a UMVUE of θ.

   (c) Note that $I[X_1 = 1]$ is an unbiased estimator of θ^{-1}. Rao-Blackwellize this using Y. Is your resulting function of Y a UMVUE of θ^{-1}?

4. (Knight) Optional (not required but recommended) Suppose that $X_1, X_2, \ldots, X_n$ are iid exponential with mean β, i.e. with marginal density
$$f(x|\beta) = I[0 < x]\, \frac{1}{\beta} \exp\left(-\frac{x}{\beta}\right)$$

Let

$$\gamma(\beta) = \exp\left(-\frac{t}{\beta}\right)$$

for some fixed number, t. ($\gamma(\beta)$ is the probability that $X_1 > t$.)

(a) Show that for every β, X_1 and $X_1 / \sum_{i=1}^{n} X_i$ are independent random variables.

(b) Rao-Blackwellize $I[X_1 > t]$ using the natural sufficient statistic here. Is the result a UMVUE of $\gamma(\beta)$? Explain.

5. Problems 3.4.12, 3.4.22 of B&D

6. (Optional only, recommended but not required) Problems 3.4.5 and 3.4.20 of B&D (Note that the "P" in 3.4.20 must refer to the joint distribution of (X, θ) specified by the likelihood and prior.)