STAT 543

HW 7 Solution

Spring 2016

#4.1.1.

(a)

$\beta(\theta) = E_\theta[\delta_c(X) = 1] = P_\theta[M_n \geq c] = 1 - P_\theta[M_n \leq c] = 1 - \left[\int_0^c \frac{1}{\theta}\,dx\right]^n = 1 - \left[\frac{c}{\theta}\right]^n.$

Since $\beta'(\theta) = -n \cdot \left[\frac{c}{\theta}\right]^{n-1} \cdot \left[-\frac{c}{\theta^2}\right] = n\frac{c^n}{\theta^{n+1}} > 0, \quad \forall \theta > 0.$

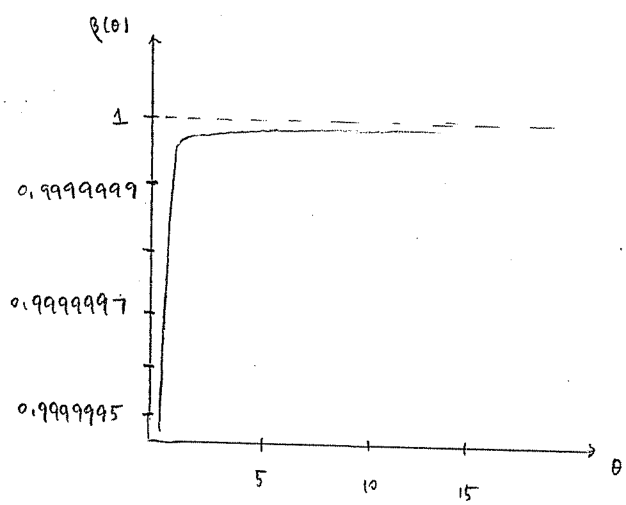
$\beta(\theta)$ is a monotone increasing fct of θ. //

(b)

$\alpha = \sup_{\Theta_0} \beta(\theta) = \beta(\frac{1}{2})$ by (a)

$= 1 - (2c)^n \overset{set}{=} 0.05 \qquad \therefore\ c = \frac{1}{2}(0.95)^{\frac{1}{n}}$ //

(c)



(d)

$$\beta(\theta) = 1 - \left[\frac{c}{\theta}\right]^n = 1 - \left[\frac{1}{2}(0.95)^{\frac{1}{n}} \cdot \frac{4}{3}\right]^n = 1 - 0.95\left(\frac{2}{3}\right)^n \overset{\text{set}}{=} 0.98$$

$$\uparrow$$

$$\left(\begin{array}{l} c = \frac{1}{2}(0.95)^{\frac{1}{n}} \\ \theta = \frac{3}{4} \end{array} \right.$$

$$\Rightarrow \quad n = \frac{\log 0.02 - \log 0.95}{\log 2 - \log 3} = 9.52 \qquad \therefore \ n = 10 \qquad //$$

(e)

$$P = \sup_{\Theta_0} (M_{20} \geq 0.48) = \sup_{\theta} \left\{ 1 - \left[\frac{0.48}{\theta}\right]^{20} : \theta \leq \frac{1}{2} \right\} = 1 - (2 \times 0.48)^{20} = 0.558$$

//

#4.1.3.

(a)

sol)

For any fixed $\theta_1 \leq \theta_0$ and $\theta_2 > \theta_0$,

$$\frac{f_{\theta_2}(\underset{\sim}{x})}{f_{\theta_1}(\underset{\sim}{x})} = \frac{\prod_{i=1}^{n} \frac{e^{-\theta_2}\theta_2^{x_i}}{x_i!}}{\prod_{i=1}^{n} \frac{e^{-\theta_1}\theta_1^{x_i}}{x_i!}} = \frac{e^{-n\theta_2} \cdot \theta_2^{\sum_1^n x_i}}{e^{-n\theta_1} \cdot \theta_1^{\sum_1^n x_i}} = e^{-n(\theta_2-\theta_1)} \cdot \left[\frac{\theta_2}{\theta_1}\right]^{\sum_1^n x_i} \triangleq g_{\theta_1,\theta_2}(T(\underset{\sim}{x}))$$

where $T(\underset{\sim}{x}) = \sum_1^n x_i$.

Since $g_{\theta_1,\theta_2}(t) = e^{-n(\theta_2-\theta_1)}\left[\frac{\theta_2}{\theta_1}\right]^t = e^{-n(\theta_2-\theta_1) + t(\log\theta_2 - \log\theta_1)}$ and

$$\frac{\partial g}{\partial t} = (\log\theta_2 - \log\theta_1)\, e^{-n(\theta_2-\theta_1) + t(\log\theta_2 - \log\theta_1)} > 0, \quad \forall t,$$

g_{θ_1,θ_2} is nondecreasing in $\sum_1^n x_i$, for $\underset{\sim}{x} \in \{\underset{\sim}{x} \in \mathcal{X} : f_{\theta_1}(\underset{\sim}{x}) + f_{\theta_2}(\underset{\sim}{x}) > 0\}$

$\therefore$ f_θ has MLR in $\sum x_i$.

Then, a UMP α-test is given by

$$\phi(x) = \begin{cases} 1 & \text{if } \sum_1^n x_i > c \\ \gamma & \text{if } \sum_1^n x_i = c \\ 0 & \text{if } \sum_1^n x_i < c \end{cases} \quad \overset{\text{equiv.}}{\Longleftrightarrow} \quad \begin{cases} 1 & \text{if } \bar{x} > \frac{c}{n} \\ \gamma & \text{if } \bar{x} = \frac{c}{n} \\ 0 & \text{if } \bar{x} < \frac{c}{n} \end{cases}$$

where c and γ are determined by

$$\alpha = \max_{\Theta_0} E_\theta \phi(x) = \max_{\Theta_0} \left[P_\theta(\bar{x} > \tfrac{c}{n}) + \gamma \cdot P_\theta(\bar{x} = \tfrac{c}{n}) \right]$$

$$= \max_{\Theta_0} \left[P_\theta(\textstyle\sum_1^n x_i > c) + \gamma \cdot P_\theta(\sum_1^n x_i = c) \right]$$

$$= \sum_{y=c+1}^{\infty} \frac{e^{-n\theta_0} \cdot (n\theta_0)^y}{y!} + \gamma \cdot \frac{e^{-n\theta_0} \cdot (n\theta_0)^c}{c!} \qquad \text{as } Y \equiv \sum_1^n x_i \sim \text{Poisson}(n\theta).$$

//

(b)

$$\beta(\theta) \equiv E_\theta \phi(x) = \sum_{y=c+1}^{\infty} \frac{e^{-n\theta}(n\theta)^y}{y!} + \gamma \cdot \frac{e^{-n\theta}(n\theta)^c}{c!}$$

$$= 1 - \sum_{y=0}^{c-1} \frac{e^{-n\theta}(n\theta)^y}{y!} + (\gamma - 1) \frac{e^{-n\theta}(n\theta)^c}{c!}$$

$$\Rightarrow \beta'(\theta) = -\sum_{y=0}^{c-1} \frac{1}{y!} \left[-ne^{-n\theta}(n\theta)^y + e^{-n\theta} \cdot y(n\theta)^{y-1} \cdot n \right] + (\gamma-1)\frac{1}{c!}\left[-ne^{-n\theta}(n\theta)^c + e^{-n\theta} c(n\theta)^{c-1} \cdot n \right]$$

$$= -\sum_{y=0}^{c-1} \frac{1}{y!} ne^{-n\theta}(n\theta)^{y-1}[y - n\theta] + \frac{\gamma - 1}{c!} ne^{-n\theta}(n\theta)^{c-1}[c - n\theta] > 0$$

for θ is not too small (as $y - n\theta < 0$ and $\gamma \in (0,1)$)

$\therefore$ $\beta(\theta)$ is increasing in θ. //

(c)

By CLT, $Z \equiv \frac{\sqrt{n}(\bar{X}-\theta_0)}{\sqrt{\theta_0}} \xrightarrow{d} N(0,1)$ as $n \to \infty$,

$$\Rightarrow \quad P(\bar{X} > \tfrac{c}{n}) = P(\bar{X} > k) = P\left(\frac{\sqrt{n}(\bar{X}-\theta_0)}{\sqrt{\theta_0}} > \frac{\sqrt{n}}{\sqrt{\theta_0}} (k-\theta_0) \right) \overset{\text{set}}{=} \alpha .$$

$\uparrow$ let $k = \frac{c}{n}$

$$\therefore \quad k = \theta_0 + \frac{\sqrt{\theta_0}}{\sqrt{n}} Z_{1-\alpha} \qquad (\Leftrightarrow c = n\theta_0 + \sqrt{n\theta_0}\, Z_{1-\alpha}) .$$

where $\Phi(Z_{1-\alpha}) = 1-\alpha$ with a standard normal cdf, Φ. //

4.2.1.

$\underset{\sim}{X} = (X_1, \cdots, X_n) \sim N(\mu, \sigma^2)$, σ^2 : known

$H_0: \mu = 0$ vs. $H_1: \mu = v$ where v is a known signal.

By N-P Lemma, we can construct the following test as

$$\delta(x) = \begin{cases} 1 & \text{if } \frac{f(z \mid H_1)}{f(z \mid H_0)} > k \\ 0 & \text{o.w.} \end{cases} \quad \text{for some } k$$

where

$$\frac{f(z \mid H_1)}{f(z \mid H_0)} = \frac{(2\pi\sigma^2)^{-\frac{n}{2}} \exp\{-\frac{1}{2\sigma^2}\sum_1^n (x_i - v)^2\}}{(2\pi\sigma^2)^{-\frac{n}{2}} \exp\{-\frac{1}{2\sigma^2}\sum_1^n x_i^2\}} = \exp\left[-\frac{1}{2\sigma^2}\{-2v\sum_1^n x_i + nv^2\}\right]$$

$$\overset{\text{equiv.}}{\Longleftrightarrow} \quad \delta(x) = \begin{cases} 1 & \text{if } \bar{X} > c \\ 0 & \text{o.w.} \end{cases} \quad \text{for some } c$$

$$\Rightarrow \quad \begin{cases} P(\delta(X)=1 \mid H_0) = P(\bar{X} > c \mid H_0) \le 0.05 \\ P(\delta(X)=0 \mid H_1) = P(\bar{X} \le c \mid H_1) \le 0.05 \end{cases}$$

$$\Rightarrow \quad \begin{cases} \frac{\sqrt{n}\,c}{\sigma} \geq Z_{1-\alpha} = 1.645 \\ \frac{\sqrt{n}(c-v)}{\sigma} \le Z_\alpha = -1.645 \end{cases}$$

$$\Rightarrow \quad 1.645 - \frac{\sqrt{n}v}{\sigma} \le \frac{\sqrt{n}c}{\sigma} - \frac{\sqrt{n}v}{\sigma} \le -1.645 \qquad \therefore \sqrt{n} \geq 3.29\frac{\sigma}{v} \Rightarrow n \geq (3.29)^2\left(\frac{\sigma}{v}\right)^2$$

① The first system: $n \geq (3.29)^2 \cdot \left(\frac{1}{2}\right)^2 = 2.7$ $\quad \therefore n=3$

② The second system: $n \geq (3.29)^2 = 10.98$ $\quad \therefore n=11$

$\Rightarrow$ ① Cost for 1st system: $10^6 + 3 \times 100 \times 10^3 = 1.3 \times 10^6$

② Cost for 2nd system: $10^5 + 11 \times 100 \times 10^3 = 1.2 \times 10^6$

$\therefore$ The 2nd system is cheaper. //

#4.2.3.

(a)

To test $H_0: \theta_1=\theta_2=\theta_3=\theta_4=0.17,\ \theta_5=0.14,\ \theta_6=0.18$ vs. $H_1: \theta_1=\theta_2=\cdots=\theta_6=\frac{1}{6}$,

we can use N-P Lemma as follows.

$$\delta(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \frac{f(\underset{\sim}{x} \mid H_1)}{f(\underset{\sim}{x} \mid H_0)} > k \\ 0 & \text{o.w.} \end{cases} \quad \text{for some } k$$

where $\dfrac{f(\underline{x}|H_1)}{f(\underline{x}|H_0)} = \dfrac{\dfrac{n!}{x_1!x_2!\cdots x_6!}\left(\dfrac{1}{6}\right)^n}{\dfrac{n!}{x_1!x_2!\cdots x_6!}(0.17)^{n-x_5-x_6}(0.14)^{x_5}(0.18)^{x_6}}$, $n=\sum_1^6 x_i$.

$= \dfrac{1}{(1.02)^{n-x_5-x_6}(0.84)^{x_5}(1.08)^{x_6}}$

//

(b)

For $n=2$,

$\Leftarrow$: $\alpha = P(\delta(\underline{x})=1|H_0) = P(\underline{x}=(0,0,0,0,2,0)|H_0) = \dfrac{2!}{2!}(0.14)^2 = 0.0196$.

$\Rightarrow$: $\alpha = P(\delta(\underline{x})=1|H_0) = P\left(\dfrac{f(\underline{x}|H_1)}{f(\underline{x}|H_0)} > k \,|H_0\right) \overset{\text{set}}{=} 0.0196$.

Note)

$(x_5, x_6, n-x_5-x_6)$	$(2,0,0)$		$(1,0,1)$		$(0,0,2)$		$\cdots$
$\dfrac{f(\underline{x}\|H_1)}{f(\underline{x}\|H_0)}$	$\dfrac{1}{0.84^2}$	$>$	$\dfrac{1}{(0.84)(1.02)}$	$>$	$\dfrac{1}{(1.02)^2}$	$>$	$\cdots$

i) For $\dfrac{1}{(0.84)(1.02)} < k < \dfrac{1}{0.84^2}$, $\alpha = P((2,0,0)|H_0) = \dfrac{2!}{2!}(0.14)^2 = 0.0196$.

ii) For $\dfrac{1}{(1.02)^2} < k < \dfrac{1}{(0.84)(1.02)}$, $\alpha = P((2,0,0)|H_0) + P((1,0,1)|H_0)$

$= \dfrac{2!}{2!}(0.14)^2 + \dfrac{2!}{1!1!}(0.14)(0.17) = 0.0672$.

$\vdots$

$\therefore$ The most powerful level 0.0196 test rejects H_0

when two 5s are obtained.

//

(c)

Under H_0, $(X_5, X_6, Y) \sim \text{Multinom}(n, 0.18, 0.14, 0.68)$ where $Y = \sum_{i=1}^{4} X_i$

$$R(\underset{\sim}{x}) \equiv \frac{f(\underset{\sim}{x} \mid H_1)}{f(\underset{\sim}{x} \mid H_0)} = \frac{1}{(1.02)^{y} (0.84)^{x_5} (1.08)^{x_6}}$$

$$\Rightarrow \log R(\underset{\sim}{x}) = -x_5 \log 0.84 - x_6 \log 1.08 - y \log 1.02$$

$$\overset{\text{let}}{=} a_1 x_5 + a_2 x_6 + a_3 y \sim N(n\mu, n\sigma^2)$$

where $a_1 = -\log 0.84$, $a_2 = -\log 1.08$, $a_3 = -\log 1.02$,

$n\mu = n\,[a_1\theta_1 + a_2\theta_2 + a_3\theta_3]$, and

$n\sigma^2 = n\,[(a_1-\mu)^2\theta_1 + (a_2-\mu)^2\theta_2 + (a_3-\mu)^2\theta_3]$

$$\Rightarrow \alpha = P(\log R(\underset{\sim}{X}) > k \mid H_0)$$

$\Rightarrow \dfrac{k - n\mu}{\sqrt{n\sigma^2}} = z_{1-\alpha}$ where $\Phi(z_{1-\alpha}) = 1-\alpha$ with a standard normal cdf, Φ.

$$\therefore\ k = n\mu + \sqrt{n\sigma^2}\, z_{1-\alpha}.$$

//

4.1.5

As H is a simple.

$\alpha(T) = \sup_{\Theta_0} P_\theta(T > k) = P_{\theta_0}(T > k) = 1 - F_T(k)$

Since T is continuous r.v., $F_T(k) \sim U(0,1)$

and $\alpha(T) = 1 - F_T(k) \sim U(0,1)$.

$\odot$ Let $U = F_T(k)$,

$P(U \le u) = P(F_T(k) \le u) = P(k \le F_T^{-1}(u)) = F_T(F_T^{-1}(u)) = u$ where $u \in (0,1)$.

$\therefore f_U(u) = 1$, $0 < u < 1$. $\therefore F_T(k) \sim U(0,1)$.

4.1.6.

By conclusion from 4.1.5, $\alpha(T_j) \overset{iid}{\sim} U(0,1)$, $j = 1, \cdots, r$

$\Rightarrow -\log \alpha(T_j) \overset{iid}{\sim} Exp(1) \overset{d}{=} Gamma(1,1)$

$\Rightarrow -2\log \alpha(T_j) \overset{iid}{\sim} Gamma(1,2)$

$\Rightarrow -2\sum_{j=1}^{r} \log \alpha(T_j) \sim Gamma(r,2) \overset{d}{=} \chi^2_{2r}$.

//

$\odot$ claim: $X \sim U(0,1) \Rightarrow -\log X \sim Exp(1)$

Let $Y = -\log X \Rightarrow X = e^{-Y}$, $|J| = e^{-Y}$

$f_Y(y) = e^{-y}$, $0 < y < \infty$

$\therefore Y = -\log X \sim Exp(1)$

#4.2.4.

(a)

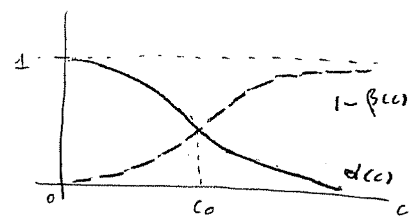
$P(\text{Type I error}) = P_{\theta_0}(L(X,\theta_0,\theta_1) \geq c) \equiv \alpha(c)$ is decreasing in c

$P(\text{Type II error}) = 1 - P_{\theta_1}(L(X,\theta_0,\theta_1) \geq c) = 1 - \beta(c)$ is increasing in c.

$\Rightarrow$ The fcts of $\alpha(c)$ and $1-\beta(c)$ will cross at c_0, i.e.,

c_0 is the solution of the equation

$$P_{\theta_0}(L(X,\theta_0,\theta_1) \geq c_0) = 1 - P_{\theta_1}(L(X,\theta_0,\theta_1) \geq c_0).$$



$$\Rightarrow \max(\alpha(c), 1-\beta(c)) = \begin{cases} \alpha(c) & \text{if } c \leq c_0 \\ 1-\beta(c) & \text{if } c \geq c_0 \end{cases}$$

$$\therefore \min_c [\max(\alpha(c), 1-\beta(c))] = \alpha(c_0)$$

$\therefore$ The likelihood ratio test with critical value c is best in this sense! //

(b)

$P_{\theta_0}(L(X,\theta_0,\theta_1) \geq c) = P_{\theta_0}\left(\frac{\sqrt{n}\bar{X}}{\sigma} > \frac{\sqrt{n}c}{\sigma}\right)$

$P_{\theta_1}(L(X,\theta_0,\theta_1) \geq c) = P_{\theta_1}\left(\frac{\sqrt{n}(\bar{X}-v)}{\sigma} > \frac{\sqrt{n}(c-v)}{\sigma}\right)$

From $P_{\theta_0}(L(X,\theta_0,\theta_1) \geq c) = 1 - P_{\theta_1}(L(X,\theta_0,\theta_1) \geq c)$, $\quad c = -(c-v) \quad \therefore c = \frac{v}{2}$

$$\therefore \delta(x) = \begin{cases} 1 & \text{if } \bar{x} > \frac{v}{2} \\ 0 & \text{if } \bar{x} \leq \frac{v}{2}. \end{cases}$$

//

4.2.5

$$H_0 : \begin{bmatrix} X \\ Y \end{bmatrix} \sim N_2\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0.6 \\ 0.6 & 1 \end{bmatrix}\right) \quad \text{vs.} \quad H_1 : \begin{bmatrix} X \\ Y \end{bmatrix} \sim N_2\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 0.6 \\ 0.6 & 1 \end{bmatrix}\right)$$

Let $T = X+Y$. Then, $T \overset{\text{Under } H_0}{\sim} N(0, 3.2)$ $\quad T \overset{\text{Under } H_1}{\sim} N(2, 3.2)$

$$\Rightarrow L(T, \theta_0, \theta_1) = \frac{e^{-\frac{1}{2(3.2)}(T-2)^2}}{e^{-\frac{1}{2(3.2)}T^2}} = e^{-\frac{1}{6.4}\{(T-2)^2 - T^2\}}$$

$$\Rightarrow \log L(T, \theta_0, \theta_1) = -\frac{1}{6.4}(T-2)^2 + \frac{1}{6.4}T^2 = \frac{T-1}{1.6}$$

$$\Rightarrow \delta(X,Y) = \begin{cases} 1 & \text{If } T \geq 1.6c+1 \\ 0 & \text{o.w.} \end{cases}$$

where $k \equiv 1.6c+1$ is determined s.t.

$$\alpha(k) = P_{\theta_0}(T \geq k) = 1 - P_{\theta_1}(T \geq k) = 1 - \beta(k)$$

$$\Rightarrow 1 - \Phi\left(\frac{k}{\sqrt{3.2}}\right) = 1 - \left[1 - \Phi\left(\frac{k-2}{\sqrt{3.2}}\right)\right]$$

$$\Rightarrow 1 - \Phi\left(\frac{k}{\sqrt{3.2}}\right) = \Phi\left(\frac{k-2}{\sqrt{3.2}}\right)$$

$$\Rightarrow k = -k+2 \qquad \therefore k = 1$$

i.e.,

$$\delta(X,Y) = \begin{cases} 1 & \text{If } T \geq 1 \\ 0 & \text{If } T < 1. \end{cases}$$

//

#3.

Since $X \sim \text{Poisson}(\lambda)$ and $\lambda \sim \text{Exp}(1)$,

$$g(\lambda|x) \propto f(x|\lambda)\cdot g(\lambda) \propto e^{-\lambda}\cdot\lambda^{x}\cdot e^{-\lambda} = \underbrace{\lambda^{x}e^{-2\lambda}}_{\text{a kernel of Gamma dist. family}} \qquad \therefore \lambda|X \sim \text{Gamma}(x+1, \tfrac{1}{2})$$

$$\Rightarrow \quad \phi(x) = \begin{cases} 1 & \text{if } P(\lambda>2|X) > 0.5 \\ 0 & \text{o.w.} \end{cases}$$

where $P(\lambda>2|X) = 1 - P(\lambda \le 2|X) = 1 - \int_0^2 \frac{1}{\Gamma(x+1)\left(\frac{1}{2}\right)^{x+1}} \lambda^{x} e^{-2\lambda}\,d\lambda$ //

#4.

Since $X \sim \text{Bin}(5,p)$ and $p \sim \text{Beta}(2,1)$,

$$g(p|x) \propto f(x|p)\,g(p) \propto p^{x}(1-p)^{5-x}\,p^{1}(1-p)^{0} = \underbrace{p^{x+1}(1-p)^{5-x}}_{\text{a kernel of Beta dist. family}}$$

$\therefore\ p|X \sim \text{Beta}(X+2,\ 6-x)$

$$\Rightarrow \quad \phi(x) = \begin{cases} 1 & \text{if } P(p<0.4)+P(p>0.6) > 0.5 \\ 0 & \text{o.w.} \end{cases}$$

where $P(p<0.4)+P(p>0.6) = 1 - \int_{0.4}^{0.6} \frac{\Gamma(8)}{\Gamma(x+2)\Gamma(6-x)} p^{x+1}(1-p)^{5-x}\,dp$ //