STAT 543

HW 8 Solution

Spring 2016

 # 4.3.1.

(a) $f(\underset{\sim}{x}|\theta) = \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{x_i}}{x_i!} = \frac{1}{\prod_{i=1}^{n} x_i!} e^{-n\theta}\theta^{\sum_1^n x_i}$.

For $\theta_2 > \theta_1$,

$$\frac{f(\underset{\sim}{x}|\theta_2)}{f(\underset{\sim}{x}|\theta_1)} = \frac{\frac{1}{\prod_{i=1}^n x_i!} e^{-n\theta_2}\theta_2^{\sum_1^n x_i}}{\frac{1}{\prod_{i=1}^n x_i!} e^{-n\theta_1}\theta_1^{\sum_1^n x_i}} = e^{-n(\theta_2-\theta_1)}\left[\frac{\theta_2}{\theta_1}\right]^{\sum_1^n x_i} \hat{=}\ g_{\theta_1,\theta_2}(T(\underset{\sim}{x})) \quad \text{where } T(\underset{\sim}{x}) = \sum_1^n x_i.$$

$\Rightarrow$ g_{θ_1,θ_2} is increasing in $T(\underset{\sim}{x}) = \sum_1^n x_i$ $\therefore$ f_θ is an MLR family in $T(\underset{\sim}{x})$.

So, a UMP test statistic is

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \sum_1^n x_i > k \\ \gamma & \text{if } \sum_1^n x_i = k \\ 0 & \text{if } \sum_1^n x_i < k \end{cases}$$ //

(b)

$\alpha = E_{\theta_0}\phi(\underset{\sim}{x}) = P_{\theta_0}(\sum_1^n x_i > k) + \gamma P_{\theta_0}(\sum_1^n x_i = k)$ for $\gamma \in [0,1]$.

where $\sum_1^n x_i \sim \text{Poisson}(n\theta_0)$.

By choosing γ approximately, any level UMP test can be exhibited. //

(c)

Note that $\sum_1^n x_i \sim \text{Poisson}(n\theta)$ $\Rightarrow$ $E\bar{X} = \frac{1}{n}\cdot E(\sum_1^n x_i) = \theta$, $Var\bar{X} = \frac{1}{n^2}Var(\sum_1^n x_i) = \frac{\theta}{n}$

Then, $\frac{\sum_1^n x_i - n\theta}{\sqrt{n\theta}} = \frac{\bar{X}-\theta}{\sqrt{\frac{\theta}{n}}} \xrightarrow{d} N(0,1)$ by CLT, $\therefore$ we can use standard normal table. //

#4.3.2.

From #4.3.1, we have the UMP test for $H_0: \theta \le \theta_0 = 10$ vs. $H_1: \theta > \theta_0 = 10$

is given by $\phi^*(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \sum_1^n x_i > k \\ 0 & \text{o.w.} \end{cases}$

where $\dfrac{\sum_1^n x_i - n\theta}{\sqrt{n\theta}} \dot\sim N(0,1)$ and this satisfies

i) $\alpha = P_{\theta_0}(\sum_1^n x_i > k) \le 0.01$

ii) $P_\theta(\sum_1^n x_i \le k \mid \theta \ge \theta_1 = 15) \le 1 - P_{\theta_1}(\sum_1^n x_i > k) \le 0.01$.

From i), $P_{\theta_0}(\sum_1^n x_i > k) \approx P_{\theta_0}\left(Z > \dfrac{k - 10n}{\sqrt{10n}}\right) \le 0.01$ as $Z = \dfrac{\sum_1^n x_i - 10n}{\sqrt{10n}} \dot\sim N(0,1)$

$\Rightarrow \dfrac{k-10n}{\sqrt{10n}} \ge z_{0.99}$ where $P(Z \le z_{0.99}) = 0.99$

$\therefore k \ge z_{0.99} \cdot \sqrt{10n} + 10n \quad \cdots ①$

From ii), $P_{\theta_1}(\sum_1^n x_i \ge k) \approx P_{\theta_1}\left(Z' \ge \dfrac{k-15n}{\sqrt{15n}}\right) \ge 0.99$ where $Z' = \dfrac{k-15n}{\sqrt{15n}} \dot\sim N(0,1)$

$\Rightarrow \dfrac{k-15n}{\sqrt{15n}} \le -z_{0.99}$

$\therefore k \le 15n - z_{0.99} \cdot \sqrt{15n} \quad \cdots ②$

By ① and ②, $z_{0.99}\sqrt{10n} + 10n \le k \le 15n - z_{0.99}\sqrt{15n}$

$\Rightarrow 5n \ge z_{0.99}(\sqrt{10n} + \sqrt{15n}) \quad \therefore n = 11.$ //

4.3.5.

For $\theta_1 > \theta_0$,

$$\frac{p(x|\theta_1)}{p(x|\theta_0)} = \frac{\prod_{i=1}^{n}\binom{n}{x_i}\theta_1^{x_i}(1-\theta_1)^{n-x_i} / [1-(1-\theta_1)^n]^n}{\prod_{i=1}^{n}\binom{n}{x_i}\theta_0^{x_i}(1-\theta_0)^{n-x_i} / [1-(1-\theta_0)^n]^n}$$

$$= \left[\frac{\theta_1}{\theta_0}\right]^{\sum_1^n x_i}\left[\frac{1-\theta_1}{1-\theta_0}\right]^{\sum_1^n (n-x_i)} \left[\frac{1-(1-\theta_0)^n}{1-(1-\theta_1)^n}\right]^n \triangleq g_{\theta_0,\theta_1}(T(\underset{\sim}{x}))$$

where $T(\underset{\sim}{x}) = \sum_1^n x_i$

$\Rightarrow$ g_{θ_0,θ_1} is increasing in $T(\underset{\sim}{x}) = \sum_1^n x_i$

$\because$ $g_{\theta_1,\theta_2}(t) = \left[\frac{\theta_1}{1-\theta_1}\cdot\frac{1-\theta_0}{\theta_0}\right]^t \cdot \left[\frac{1-\theta_1}{1-\theta_0}\right]^{n^2}\left[\frac{1-(1-\theta_0)^n}{1-(1-\theta_1)^n}\right]^n$

$$\Rightarrow \frac{\partial g}{\partial t} = \left[\frac{1-\theta_1}{1-\theta_0}\right]^{n^2}\left[\frac{1-(1-\theta_0)^n}{1-(1-\theta_1)^n}\right]^n \cdot \{\log\theta_1 - \log\theta_0 + \log(1-\theta_0) - \log(1-\theta_1)\}\, e^{t\log\frac{\theta_1}{1-\theta_1}\cdot\frac{1-\theta_0}{\theta_0}} > 0$$

$\therefore$ $p(x|\theta)$ has MLR in $T(\underset{\sim}{x}) = \sum_1^n x_i$.

Thus, $T(\underset{\sim}{x}) = \sum_1^n x_i$ is an optimal test statistic for $H_0: \theta = \theta_0$ vs. $H_1: \theta > \theta_0$. //

4.3.6

(a)

$$\mu \equiv EX = \int_c^\infty x\cdot c^\theta \theta x^{-(1+\theta)}dx = c^\theta\cdot\theta\cdot\int_c^\infty x^{-\theta}dx = c^\theta\cdot\theta\cdot\frac{1}{-\theta+1}x^{-\theta+1}\Big|_c^\infty = \frac{\theta}{\theta-1}c.$$ //

(b)

Since $\mu = \frac{\theta}{\theta-1}c \Leftrightarrow \theta = \frac{\mu}{\mu-c}$, $\quad f(x,\mu) = c^{\frac{\mu}{\mu-c}}\cdot\frac{\mu}{\mu-c}x^{-(1+\frac{\mu}{\mu-c})}$, $x > c$.

For $\mu_1 > \mu_0$,

$$\frac{f(\underline{x},\mu_1)}{f(\underline{x},\mu_0)} = \frac{c^{\frac{n\mu_1}{\mu_1-c}}\left[\frac{\mu_1}{\mu_1-c}\right]^n\left[\prod_{i=1}^n x_i\right]^{-(1+\frac{\mu_1}{\mu_1-c})}}{c^{\frac{n\mu_0}{\mu_0-c}}\left[\frac{\mu_0}{\mu_0-c}\right]^n\left[\prod_{i=1}^n x_i\right]^{-(1+\frac{\mu_0}{\mu_0-c})}} = c^{n\left(\frac{\mu_1}{\mu_1-c}-\frac{\mu_0}{\mu_0-c}\right)}\left[\frac{\mu_1(\mu_0-c)}{\mu_0(\mu_1-c)}\right]^n$$

$$\times \exp\left\{\left(\sum_1^n \log x_i\right)\left(\frac{\mu_0}{\mu_0-c}-\frac{\mu_1}{\mu_1-c}\right)\right\} \triangleq g_{\mu_0,\mu_1}(T(\underline{x}))$$

where $T(\underline{x}) = \sum_1^n \log x_i$.

Since
$$\frac{\mu_0}{\mu_0-c} - \frac{\mu_1}{\mu_1-c} = \frac{(\mu_1-\mu_0)c}{(\mu_0-c)(\mu_1-c)} > 0 \quad \text{as} \quad \mu_1>\mu_0,$$

g_{μ_0,μ_1} is increasing in $T(\underline{x}) = \sum_1^n \log x_i$ and thus, $f(\underline{x},\mu)$ is a MLR family in $T(\underline{x}) = \sum_1^n \log x_i$.

$\therefore$ $T(\underline{x}) = \sum_1^n \log x_i$ is an optimal test statistic for $H: \mu = \mu_0$ vs. $K: \mu > \mu_0$. //

(c)

$$\eta \equiv -\theta$$

$$f(\underline{x},\theta) = \prod_{i=1}^n \frac{1}{x_i}\cdot \exp\left[-\theta\sum_1^n \log x_i + n\theta\log c + n\log\theta\right] = \prod_{i=1}^n \frac{1}{x_i!}\exp\left[\eta\sum_1^n \log x_i - n\eta\log c + n\log(-\eta)\right]$$

$\Rightarrow$ $T(\underline{x}) \equiv \sum_1^n \log x_i$ $\qquad A(\eta) = n\eta\log c - n\log(-\eta)$

By Thm 1.6.2,

$$E\left[\sum_1^n \log x_i\right] = n\log c - \frac{n}{\eta} = n\log c + \frac{n}{\theta}$$

$$V\left[\sum_1^n \log x_i\right] = \frac{n}{\eta^2} = \frac{n}{\theta^2}$$

By CLT,
$$\frac{\frac{1}{n}\sum_1^n \log x_i - (\log c + \frac{1}{\theta})}{\sqrt{\frac{1}{n\theta^2}}} \xrightarrow{d} N(0,1),$$
the critical value of the test in part (b) is $n\log c + \frac{n}{\theta} + z_{1-\alpha}\sqrt{\frac{n}{\theta^2}}$ //

#4.3.9.

For $\theta_1 > \theta_0 = 1$,

$$\lambda = \frac{f(\underset{\sim}{x}, \theta_1)}{f(\underset{\sim}{x}, \theta_0)} = \frac{\theta_1^n \left[\prod_{i=1}^n x_i\right]^{\theta_1 - 1} \exp\left(-\sum_1^n x_i^{\theta_1}\right)}{\exp\left(-\sum_1^n x_i\right)} = \theta_1^n \left[\prod_{i=1}^n x_i\right]^{\theta_1} \exp\left[-\sum_1^n x_i^{\theta_1} + \sum_1^n x_i\right]$$

For any $\theta_1 > 1$, by NP Thm, $\exists$ an MP test for testing $H: \theta = 1$ vs. $K: \theta = \theta_1$ in the form of

$$\phi_1(\underset{\sim}{x}) = \begin{cases} 1 & \dfrac{f(\underset{\sim}{x}, \theta_1)}{f(\underset{\sim}{x}, \theta_0)} \geq k_1 \\ 0 & \text{o.w.} \end{cases}$$

Then, for any other $\theta_2 > 1$, $\theta_2 \neq \theta_1$, $\exists$ also an MP test as

$$\phi_2(\underset{\sim}{x}) = \begin{cases} 1 & \dfrac{f(\underset{\sim}{x}, \theta_2)}{f(\underset{\sim}{x}, \theta_0)} \geq k_2 \\ 0 & \text{o.w.} \end{cases}$$

Since both likelihood ratio test stat. have θ,

they cannot be MP for the two tests, simultaneously.

$\therefore$ The MP test for $H: \theta = 1$ vs. $K: \theta = \theta_1 > 1$ is not UMP. //

#4.3.11.

The optimal Bayes test for $H: \theta \le \theta_0$ vs. $K: \theta > \theta_1$ is,

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \dfrac{P(\theta > \theta_1 | \underset{\sim}{x})}{P(\theta \le \theta_0 | \underset{\sim}{x})} > 1 \\ 0 & \text{o.w.} \end{cases}$$

Since $$\frac{P(\theta > \theta_1 | \underset{\sim}{x})}{P(\theta \le \theta_0 | \underset{\sim}{x})} = \frac{\int_{\theta_1}^{\infty} p(\underset{\sim}{x}, \theta) \, d\pi(\theta)}{\int_{-\infty}^{\theta_0} p(\underset{\sim}{x}, \theta) \, d\pi(\theta)} = \frac{\int_{\theta_1}^{\infty} L(\underset{\sim}{x}, \theta, \theta_0) \, d\pi(\theta)}{\int_{-\infty}^{\theta_0} L(\underset{\sim}{x}, \theta_1, \theta_0) \, d\pi(\theta)}$$ by hmt.

where $\int_{\theta_1}^{\infty} L(\underset{\sim}{x}, \theta, \theta_0) \, d\pi(\theta)$ is increasing in $T(\underset{\sim}{x})$, and

$\int_{-\infty}^{\theta_0} L(\underset{\sim}{x}, \theta, \theta_0) \, d\pi(\theta)$ is decreasing in $T(\underset{\sim}{x})$,

$$\frac{\int_{\theta_1}^{\infty} L(\underset{\sim}{x}, \theta, \theta_0) \, d\pi(\theta)}{\int_{-\infty}^{\theta_0} L(\underset{\sim}{x}, \theta, \theta_0) \, d\pi(\theta)}$$ is increasing in $T(\underset{\sim}{x})$.

$\therefore$ $\phi(\underset{\sim}{x})$ is equivalent to $$\phi^*(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } T(\underset{\sim}{x}) > k \\ 0 & \text{o.w.} \end{cases}$$ //.

P2.

For $\alpha_1 > \alpha_0$ where $\alpha_0, \alpha_1 \in [0,1]$,

$$\frac{g_{\alpha_1}}{g_{\alpha_0}} = \frac{\alpha_1 g_1 + (1-\alpha_1) g_0}{\alpha_0 g_1 + (1-\alpha_0) g_0} = \frac{\alpha_1 \frac{g_1}{g_0} + 1-\alpha_1}{\alpha_0 \frac{g_1}{g_0} + 1-\alpha_0} \hat{=} g_{\alpha_0,\alpha_1}(T)$$

$$\Rightarrow \frac{\partial g}{\partial T} = \frac{\alpha_1 \cdot \frac{d}{dT}\left(\frac{g_1}{g_0}\right)\left[\alpha_0 \frac{g_1}{g_0} + 1-\alpha_0\right] - \left[\alpha_1 \frac{g_1}{g_0} + 1-\alpha_1\right] \cdot \alpha_0 \frac{d}{dT}\left(\frac{g_1}{g_0}\right)}{\left(\alpha_0 \frac{g_1}{g_0} + 1-\alpha_0\right)^2}$$

$$= \frac{[\alpha_1(1-\alpha_0) - \alpha_0(1-\alpha_1)] \frac{d}{dT}\left(\frac{g_1}{g_0}\right)}{\left(\alpha_0 \frac{g_1}{g_0} + 1-\alpha_0\right)^2} = \frac{(\alpha_1-\alpha_0) \frac{d}{dT}\left(\frac{g_1}{g_0}\right)}{\left(\alpha_0 \frac{g_1}{g_0} + 1-\alpha_0\right)^2} \geq 0,$$

since $\alpha_1 > \alpha_0$ and $\frac{d}{dT}\left(\frac{g_1}{g_0}\right) \geq 0$ as $\frac{g_1}{g_0}$ is nondecreasing in T, $\frac{\partial g}{\partial T} \geq 0$.

$\therefore$ g_α has MLR in $T(x)$. //

P3.

pf) Suppose $\exists$ ϕ satisfying Definition 1 but not Definition 2.

i.e., $\exists$ a test ϕ' with $\alpha' \equiv \sup_{\Theta_0} \pi_{\phi'}(\theta) < \alpha$ s.t., for some $\theta^* \in \Theta_1$,

$$\pi_{\phi'}(\theta^*) > \pi_\phi(\theta^*).$$

Consider $\phi''(x) = \frac{1-\alpha}{1-\alpha'} \phi'(x) + \left(1 - \frac{1-\alpha}{1-\alpha'}\right)$

$$\Rightarrow \alpha'' \equiv \sup_{\Theta_0} \pi_{\phi''}(\theta) = \frac{1-\alpha}{1-\alpha'} \alpha' + 1 - \frac{1-\alpha}{1-\alpha'} = \alpha.$$

$$\Rightarrow \pi_{\phi''}(\theta^*) = P_{\theta^*}(\phi''(x) = 1) = P_{\theta^*}\left(\frac{1-\alpha}{1-\alpha'}\phi'(x) + 1 - \frac{1-\alpha}{1-\alpha'} = 1\right) = P_{\theta^*}(\phi'(x) = 1)$$

$$= \pi_{\phi'}(\theta^*) > \pi_\phi(\theta^*) \quad \text{※} \quad \text{since } \phi'' \text{ is a test of size } \alpha.$$

//

4.9.1.

$$L(\theta) = f(x;\theta) = \binom{n}{x} \theta^{x} (1-\theta)^{n-x}$$

$$\Rightarrow \quad l(\theta) \equiv \log L(\theta) = \log \binom{n}{x} + x \log \theta + (n-x) \log (1-\theta).$$

$$\Rightarrow \quad l'(\theta) = \frac{x}{\theta} - \frac{n-x}{1-\theta} \overset{set}{=} 0 \qquad \therefore \hat{\theta} = \frac{x}{n}$$

$$l''(\theta) = -\frac{x}{\theta^2} - \frac{(n-x)}{(1-\theta)^2} < 0, \quad \forall \theta, \qquad \hat{\theta}_{MLE} = \frac{x}{n}.$$

$$\Rightarrow \quad \lambda(x) = \frac{\sup_{\Theta} L(\theta)}{\sup_{\Theta_0} L(\theta)} = \frac{\binom{n}{x}\left(\frac{x}{n}\right)^{x}\left(1-\frac{x}{n}\right)^{n-x}}{\binom{n}{x}\left(\frac{1}{2}\right)^{n}} = \left(\frac{2}{n}\right)^{n} x^{x} (n-x)^{n-x}.$$

$$\Rightarrow \quad \log \lambda(x) = n \log \frac{2}{n} + x \log x + (n-x) \log (n-x).$$

Let $t = x - \frac{n}{2}$

$$\Rightarrow \quad \log \lambda(t) = n \log \frac{2}{n} + \left(t+\frac{n}{2}\right) \log \left(t+\frac{n}{2}\right) + \left(\frac{n}{2}-t\right) \log \left(\frac{n}{2}-t\right)$$

$$\Rightarrow \quad \frac{d \log \lambda(t)}{dt} = \log \left(t+\frac{n}{2}\right) + 1 - \log \left(\frac{n}{2}-t\right) - 1 = \log \left(t+\frac{n}{2}\right) - \log \left(t-\frac{n}{2}\right)$$

where $t+\frac{n}{2} > 0$ and $\frac{n}{2} - t > 0 \Rightarrow -\frac{n}{2} < t < \frac{n}{2}$.

$$\begin{cases} \text{For } t > 0, & t+\frac{n}{2} > \frac{n}{2} - t \Rightarrow \log \lambda(t) \text{ is increasing in } t \\ \text{For } t < 0, & t+\frac{n}{2} < \frac{n}{2} - t \Rightarrow \log \lambda(t) \text{ is decreasing in } t, \end{cases}$$

which means that this is increasing in $-t$.

$\therefore \log \lambda(t)$ is increasing in $|t| \Leftrightarrow \lambda(t)$ is increasing in $|t|$.

$\Leftrightarrow \lambda(x)$ is increasing in $|2x-n|$ //

4.9.7.

(a)

$$L(\mu,\sigma^2) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_1^n (x_i-\mu)^2\right\}.$$

$$\Rightarrow \ l(\mu,\sigma^2) \equiv \log L(\mu,\sigma^2) = -\frac{n}{2}\log 2\pi\sigma^2 - \frac{1}{2\sigma^2}\sum_1^n (x_i-\mu)^2.$$

$$\Rightarrow \begin{cases} \dfrac{\partial l}{\partial \mu} = \dfrac{1}{\sigma^2}\displaystyle\sum_1^n (x_i-\mu)^2 \overset{\text{set}}{=} 0 \\[2ex] \dfrac{\partial l}{\partial \sigma^2} = -\dfrac{n}{2\sigma^2} + \dfrac{1}{2\sigma^4}\displaystyle\sum_1^n (x_i-\mu)^2 \overset{\text{set}}{=} 0 \end{cases} \Rightarrow \hat{\mu}_{MLE} = \bar{x}, \quad \hat{\sigma}^2_{MLE} = \frac{1}{n}\sum_1^n (x_i-\bar{x})^2.$$

b)

$$\lambda(x) = \frac{\sup_{\Theta} L(\mu,\sigma^2)}{\sup_{\Theta_0} L(\mu,\sigma^2)} = \begin{cases} 1 & \text{if } \hat{\sigma}^2_{MLE}/\sigma_0^2 \le 1 \\[2ex] \dfrac{(2\pi\hat{\sigma}^2_{MLE})^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\hat{\sigma}^2_{MLE}}\sum_1^n (x_i-\bar{x})^2\right\}}{(2\pi\sigma_0^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma_0^2}\sum_1^n (x_i-\bar{x})^2\right\}} & \text{if } \hat{\sigma}^2_{MLE}/\sigma_0^2 > 1. \end{cases}$$

$$= \begin{cases} 1 & \text{if } \hat{\sigma}^2_{MLE}/\sigma_0^2 \le 1 \\[2ex] \left[\dfrac{\hat{\sigma}^2_{MLE}}{\sigma_0^2}\right]^{-\frac{n}{2}} \exp\left\{\dfrac{n}{2}\left(\dfrac{\hat{\sigma}^2_{MLE}}{\sigma_0^2}-1\right)\right\} & \text{if } \hat{\sigma}^2_{MLE}/\sigma_0^2 > 1. \end{cases}$$

$$\Rightarrow \log \lambda(x) = \begin{cases} 0 & \text{if } \hat{\sigma}^2_{MLE}/\sigma_0^2 \le 1 \\[2ex] \dfrac{n}{2}\left[-\log\dfrac{\hat{\sigma}^2_{MLE}}{\sigma_0^2} - 1 + \dfrac{\hat{\sigma}^2_{MLE}}{\sigma_0^2}\right] & \text{if } \hat{\sigma}^2_{MLE}/\sigma_0^2 > 1. \end{cases}$$

which is increasing in $\dfrac{\hat{\sigma}^2_{MLE}}{\sigma_0^2}$. $\Rightarrow$ $\lambda(x)$ is increasing in $\dfrac{\sum_1^n (x_i-\bar{x})^2}{\sigma_0^2}$.

$\therefore$ Reject iff $\dfrac{n\hat{\sigma}^2}{\sigma_0^2} = \dfrac{1}{\sigma_0^2}\displaystyle\sum_1^n (x_i-\bar{x})^2 \ge c.$ //

(b)

By thm B.3.3, $\frac{n\hat{\sigma}^2_{MLE}}{\sigma_0^2} = \frac{1}{\sigma_0^2}\sum_1^n (x_i - \bar{x})^2 \overset{H}{\sim} \chi^2_{n-1}$.

Then, $\alpha = P_{\sigma_0^2}\left(\frac{1}{\sigma_0^2}\sum_1^n (x_i-\bar{x})^2 \geq c\right)$

$\Leftrightarrow \quad 1-\alpha = P_{\sigma_0^2}\left(\frac{1}{\sigma_0^2}\sum_1^n (x_i-\bar{x})^2 < c\right) \qquad \therefore \; c = \chi^2_{n-1}(1-\alpha).$ //

(c)

By the MLR property, this test is UMP test.

And by the duality thm, this test form a $(1-\alpha)$ confidence set for σ^2.

So, by the MLR, this test is equivalent to the test obtained by inverting the family of level $(1-\alpha)$ lower confidence bounds for σ^2. //

4.9.9

(a)

From $x_i = \theta x_{i-1} + \varepsilon_i$ where $\varepsilon_i \overset{iid}{\sim} N(0,\sigma^2)$ and $x_0 = 0$ for $i=1,\cdots,n$,

$X_i \mid x_0, \cdots, x_{i-1} \sim N(\theta x_{i-1}, \sigma^2) \quad \cdots (*)$

$\Rightarrow \; p(\underset{\sim}{x} \mid \theta) = p(x_0)\, p(x_1|x_0)\, p(x_2|x_0, x_1) \cdots p(x_n|x_0, x_1, \cdots, x_{n-1})$

$= p(x_0)\, p(x_1|x_0)\, p(x_2|x_1) \cdots p(x_n|x_{n-1})$ by $(*)$

$= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \theta x_{i-1})^2\right]$ //

(b)

i) $l(\theta) \equiv \log L(\theta) = -\frac{n}{2}\log 2\pi\sigma^2 - \frac{1}{2\sigma^2}\sum_1^n (x_i - \theta x_{i-1})^2$

$$\frac{\partial l}{\partial \theta} = \frac{1}{\sigma^2}\sum_1^n (x_i - \theta x_{i-1})\, x_{i-1} \overset{\text{set}}{=} 0$$

Since $x_0 = 0$, $\sum_{i=2}^n (x_i - \theta x_{i-1})\cdot x_{i-1} = 0 \Rightarrow \sum_{i=2}^n x_i x_{i-1} = \hat{\theta}\cdot \sum_{i=2}^n x_{i-1}^2$,

$$\therefore \hat{\theta}_{MLE} = \frac{\sum_{i=2}^n x_i x_{i-1}}{\sum_{i=2}^n x_{i-1}^2} = \frac{\sum_{i=2}^n x_i \cdot x_{i-1}}{\sum_{i=1}^{n-1} x_i^2} \equiv \hat{\theta}.$$

$\Rightarrow$ For any fixed $\sigma^2 > 0$,

$$\lambda(x) = \frac{\sup_{\Theta} L(\theta)}{\sup_{\Theta_0} L(\theta)} = \frac{(2\pi\sigma^2)^{-\frac{n}{2}}\exp\{-\frac{1}{2\sigma^2}\sum_1^n (x_i - \hat{\theta}_{MLE}x_{i-1})^2\}}{(2\pi\sigma^2)^{-\frac{n}{2}}\exp\{-\frac{1}{2\sigma^2}\sum_1^n x_i^2\}}$$

$$= \exp\{-\frac{1}{2\sigma^2}\sum_1^n (\hat{\theta}_{MLE}^2 x_{i-1}^2 - 2\hat{\theta}_{MLE}x_{i-1}x_i)\} = \exp\{\frac{1}{2\sigma^2}\frac{\left(\sum_{i=2}^n x_{i-1}x_i\right)^2}{\sum_{i=1}^{n-1} x_i^2}\}$$

$\Rightarrow$ $\lambda(x)$ is increasing in $\dfrac{\left(\sum_{i=2}^n x_i x_{i-1}\right)^2}{\sum_{i=1}^{n-1} x_i^2}$

$\therefore$ LR test statistic is equivalent to $\dfrac{\left(\sum_{i=2}^n x_i x_{i-1}\right)^2}{\sum_{i=1}^{n-1} x_i^2}$. //

#4.9.10.

(a)

Note that $0<\alpha<\frac{1}{2}$ and, $\frac{\alpha}{2(1-\alpha)}<c<\alpha \Rightarrow 1<\frac{1-c}{1-\alpha}<\frac{1}{1-\alpha}<\frac{2c}{\alpha}$.

Then,

x	-2	-1	0	1	2
$L(-1)$	$\frac{\alpha}{2}$	$\frac{1}{2}-\alpha$	α	$\frac{1}{2}-\alpha$	$\frac{\alpha}{2}$
$\sup_{\Theta} L(\theta)$	c	$\left(\frac{1-c}{1-\alpha}\right)\left(\frac{1}{2}-\alpha\right)$	$\left(\frac{1-c}{1-\alpha}\right)\alpha$	$\left(\frac{1-c}{1-\alpha}\right)\left(\frac{1}{2}-\alpha\right)$	c
$\lambda(x)=\frac{\sup_{\Theta} L(\theta)}{L(-1)}$	$\frac{2c}{\alpha}$	$\frac{1-c}{1-\alpha}$	$\frac{1-c}{1-\alpha}$	$\frac{1-c}{1-\alpha}$	$\frac{2c}{\alpha}$

$\Rightarrow \alpha = P\left(\lambda(x) > \frac{1-c}{1-\alpha} \mid \theta=-1\right) = P(X=\pm 2 \mid \theta=-1)$

So, the size α LRT for testing $H: \theta=-1$ vs. $K: \theta\neq -1$ is

$$\phi(x) = \begin{cases} 1 & \text{if } x=\pm 2 \\ 0 & \text{o.w.} \end{cases}$$ //

(b)

Consider a test of $\phi^*(x) = \begin{cases} 1 & \text{if } x=0 \\ 0 & \text{o.w.} \end{cases}$

① Letting $\theta_0=-1$, $E_{\theta_0}\phi^*(x) = P_{\theta_0}(x=0) = \alpha$

② For any $\theta_1 \in [0,1]$,

$E_{\theta_1}\phi^*(x) = P(x=0 \mid \theta=\theta_1) = \left(\frac{1-c}{1-\alpha}\right)\alpha$ and $E_{\theta_1}\phi(x) = P(x \pm 2 \mid \theta=\theta_1) = c$.

Since $\left(\frac{1-c}{1-\alpha}\right)\alpha - c = \frac{\alpha - c}{1-\alpha} > 0$, ϕ^* is strictly more powerful for $\forall \theta \in [0,1]$.

//

#4.7.2.

(a)

$$T = \sum_{1}^{n} X_i \sim Po(n\lambda)$$

$$\lambda = \frac{V}{S_0} \Rightarrow S_0 \lambda = V \sim \chi^2_k.$$

$$\Rightarrow p(\lambda | T = t) \propto P(T = t | \lambda)\, g(\lambda | S_0) \propto e^{-\lambda (S_0 + 2n)/2} \cdot \lambda^{\frac{k+2t}{2} - 1}$$

Let $W = (2n + S_0)\lambda \Rightarrow \lambda = \dfrac{W}{2n + S_0}$

$$f(w) \propto e^{-\frac{w}{2}} \cdot w^{\frac{2t+k}{2} - 1} \sim \chi^2_{(2t+k)}$$

$\Rightarrow \lambda | T \overset{d}{=} \dfrac{W}{S}$ where $S = S_0 + 2n$ and $W \sim \chi^2_m$ with $m = 2t + k$. //

(b)

$$1 - \alpha \overset{set}{=} P(l \le \lambda \le u) = P\left(l \le \frac{W}{S} \le u\right) = P(Sl \le W \le Su).$$

$\Rightarrow Sl = \chi^2_{m, \frac{\alpha}{2}}$ and $Su = \chi^2_{m, 1-\frac{\alpha}{2}}$

$$\therefore l = \frac{\chi^2_{m, \frac{\alpha}{2}}}{S}, \quad u = \frac{\chi^2_{m, 1-\frac{\alpha}{2}}}{S} \quad //$$

#4.4.1.

(a)

Since $Y \equiv \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sigma^2} \sim \chi^2_{n-1}$,

$$1-\alpha = P\left(\chi^2_{n-1,\frac{\alpha}{2}} \le Y \le \chi^2_{n-1,1-\frac{\alpha}{2}}\right) = P\left(\log\frac{\sum_1^n(x_i-\bar{x})^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}} \le \log\sigma^2 \le \log\frac{\sum_1^n(x_i-\bar{x})^2}{\chi^2_{n-1,\frac{\alpha}{2}}}\right)$$

$\therefore$ The $(1-\alpha)$ C.I. for $\log\sigma^2$ is $\left(\log\frac{\sum_1^n(x_i-\bar{x})^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}},\ \log\frac{\sum_1^n(x_i-\bar{x})^2}{\chi^2_{n-1,\frac{\alpha}{2}}}\right)$. //

(b)

From part (a), $(1-\alpha)$ UCB for σ^2 is $\frac{\sum_1^n(x_i-\bar{x})^2}{\chi^2_{n-1,0.005}} = 420672.8$, which is too large.

We can consider the $(1-\alpha)$ C.I. for σ^2 is $\left(0,\ \frac{\sum_1^n(x_i-\bar{x})^2}{\chi^2_{n-1,\alpha}}\right)$

and the $(1-\alpha)$ UCB for σ^2 is $\frac{\sum_1^n(x_i-\bar{x})^2}{\chi^2_{n-1,0.01}} = 105164.1$

which is better. //

#4.4.2.

(a)

$$\frac{2\sum_{i=1}^{n}t_i^2x_i}{\sum_{i=1}^{n}t_i^4} - \theta = \frac{2\sum_{i=1}^{n}t_i^2\varepsilon_i}{\sum_{i=1}^{n}t_i^4} \sim N\left(0,\ \frac{4\sigma^2}{\sum_{i=1}^{n}t_i^4}\right),$$

the level $(1-\alpha)$ C.I. for θ is $\left(\frac{2\sum_{i=1}^{n}t_i^2x_i}{\sum_{i=1}^{n}t_i^4} - z_{1-\frac{\alpha}{2}}\frac{2\sigma}{\sqrt{\sum_{i=1}^{n}t_i^4}},\ \frac{2\sum_{i=1}^{n}t_i^2x_i}{\sum_{i=1}^{n}t_i^4} + z_{1-\frac{\alpha}{2}}\frac{2\sigma}{\sqrt{\sum_{i=1}^{n}t_i^4}}\right)$

(b)

For $0 \le t_i \le 1$, $i=1,\cdots,n$,

the length of the interval $= 2 \times \dfrac{2\sigma}{\sqrt{\sum_{i=1}^{n} t_i^4}} \cdot z_{1-\frac{\alpha}{2}}$

$\geq 2 \cdot \dfrac{2\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}$ where $\sum_{i=1}^{n} t_i^4 = n$ if $\forall t_i = 1$.

$\therefore$ when $t_1 = t_2 = \cdots = t_n = 1$, the interval is shortest for given α. //

4.8.1

(a)

Since $\dfrac{x_{n+1} - \bar{x}}{\sqrt{1+\frac{1}{n}}\,\sigma_0} \sim N(0,1)$, the level $(1-\alpha)$ prediction interval for x_{n+1}

is $\left(\bar{x} - z_{1-\frac{\alpha}{2}}\sqrt{1+\frac{1}{n}}\cdot\sigma_0,\ \bar{x} + z_{1-\frac{\alpha}{2}}\sqrt{1+\frac{1}{n}}\cdot\sigma_0\right)$ //

(b)

The Bayesian prediction interval for x_{n+1} is given by

$\left(\hat{\mu}_B - z_{1-\frac{\alpha}{2}}\sqrt{\sigma_0^2 + \hat{\sigma}_B^2},\ \hat{\mu}_B + z_{1-\frac{\alpha}{2}}\sqrt{\sigma_0^2 + \hat{\sigma}_B^2}\right)$

where $\hat{\mu}_B = \dfrac{\hat{\sigma}_B^2}{\tau^2}\eta_0 + \dfrac{n\hat{\sigma}_B^2}{\sigma_0^2}\bar{x}$. and $\hat{\sigma}_B^2 = \dfrac{1}{\frac{n}{\sigma_0^2} + \frac{1}{\tau^2}}$.

* The simulation results :

μ	5	8	9	9.5	10	10.5	11	12	15
Coverage prob.	0.9514	0.9506	0.9469	0.9537	0.9454	0.9502	0.9469	0.9514	0.9507.

4.8.7.

$X_1, \cdots, X_n \overset{iid}{\sim} Exp(\theta)$.

Since $Y = \frac{2X}{\theta}$ where $X \sim Exp(\theta)$ $f_Y(y) = \frac{1}{\theta} \cdot \exp(-\frac{y}{2}) \cdot \frac{\theta}{2} = \frac{1}{2}\exp(-\frac{y}{2})$

$\Rightarrow$ letting $Y_i = \frac{2X_i}{\theta}$, $i = 1, \cdots, n$,

$$Y_i \overset{iid}{\sim} Exp(2) \overset{d}{=} Gamma(1, 2) \overset{d}{=} \chi^2_2.$$

$\Rightarrow$ $\sum_{i=1}^{n} Y_i \sim \chi^2_{2n}$, $Y_{n+1} = \frac{2X_{n+1}}{\theta} \sim \chi^2_2$ and $\sum_{i=1}^{n} Y_i \perp Y_{n+1}$

$$\Rightarrow \frac{Y_{n+1}/2}{\sum_{i=1}^{n} Y_i/2n} = \frac{n X_{n+1}}{\sum_{i=1}^{n} X_i} \sim F_{2,2n}.$$

$\therefore$ A level $(1-\alpha)$ prediction interval for X_{n+1} is

$$\left(F_{2,2n,\frac{\alpha}{2}} \cdot \left(\frac{\sum_{i=1}^{n} X_i}{n}\right),\ F_{2,2n,1-\frac{\alpha}{2}} \left(\frac{\sum_{i=1}^{n} X_i}{n}\right) \right). \quad //$$

4.5.1

(a)

From the results of Problem B.2.4 and B.2.5, $\theta X_i \sim \chi^2_2$, and $\lambda Y_i \sim \chi^2_2$.

$\Rightarrow$ $2\theta \sum_{i=1}^{n_1} X_i \sim \chi^2_{2n_1}$, $2\lambda \sum_{i=1}^{n_2} Y_i \sim \chi^2_{2n_2}$, and $\{X_i\} \perp \{Y_i\}$.

$$\Rightarrow \frac{2\theta \sum_{i=1}^{n_1} X_i / 2n_1}{2\lambda \sum_{i=1}^{n_2} Y_i / 2n_2} = \frac{\bar{X}}{\bar{Y}} \cdot \frac{\theta}{\lambda} \sim F_{2n_1, 2n_2}.$$

$\therefore P\left(\frac{\bar{Y}}{\bar{X}} f(\frac{\alpha}{2}) \le \frac{\theta}{\lambda} \le \frac{\bar{Y}}{\bar{X}} f(1-\frac{\alpha}{2})\right) = P\left(f(\frac{\alpha}{2}) \le \frac{\bar{X}}{\bar{Y}} \cdot \frac{\theta}{\lambda} \le f(1-\frac{\alpha}{2})\right) = 1-\alpha$ as $\frac{\bar{X}}{\bar{Y}} \cdot \frac{\theta}{\lambda} \sim F_{2n_1, 2n_2}$. //

(b)

From the result of (a), $P\left(\frac{\bar{Y}}{\bar{X}} f\left(\frac{\alpha}{2}\right) \le \Delta \le \frac{\bar{Y}}{\bar{X}} f\left(1-\frac{\alpha}{2}\right)\right) = 1-\alpha$,

So, the test with acceptance region $\left[f\left(\frac{\alpha}{2}\right) \le \frac{\bar{X}}{\bar{Y}} \le f\left(1-\frac{\alpha}{2}\right)\right]$ has size α

for testing $H: \Delta = 1$ vs. $K: \Delta \ne 1$

//

(c) $(0.2398,\ 1.0586)$.

$\Rightarrow$ Since the interval contains 1, we cannot reject $H: \Delta = 1$ at the level of

$\alpha = 0.1$. //

4.4.7

By theorem B.3.3, S_{n_0} is indep. of $\bar{X}_{n_0}$.

where $V \equiv \dfrac{\sqrt{N}(\bar{X}-\mu)}{\sigma} \sim N(0,1)$, $W \equiv \dfrac{(n_0-1)S_{n_0}^2}{\sigma^2} \sim \chi^2_{n_0-1}$. and $V \perp W$

$$\Rightarrow \frac{V}{\sqrt{W/(n_0-1)}} = \frac{\dfrac{\sqrt{N}(\bar{X}-\mu)}{\sigma}}{\sqrt{\dfrac{(n_0-1)S_{n_0}^2}{\sigma^2} \Big/ (n_0-1)}} = \frac{\sqrt{N}(\bar{X}-\mu)}{S_{n_0}} \sim t_{n_0-1}.$$

$\therefore$ So, $(1-\alpha)$ C.I. for μ of length at most $2d$ is

$$\left(\bar{X} - t_{n_0-1,\left(1-\frac{\alpha}{2}\right)} \cdot \frac{S_0}{\sqrt{N}},\quad \bar{X} + t_{n_0-1,\left(1-\frac{\alpha}{2}\right)} \cdot \frac{S_0}{\sqrt{N}}\right).$$

//

4.5.4

(a)

$\beta(\theta) = P_\theta(\delta_c(x)=1) = 1-\left(\frac{c}{\theta}\right)^n$, which is monotone increasing in θ.

$\Rightarrow \alpha = \sup_{\theta \le \theta_0} \beta(\theta) = 1-\left(\frac{c}{\theta_0}\right)^n \qquad \therefore c = (1-\alpha)^{\frac{1}{n}}\theta_0$. //

(b)

$$\delta_c(x) = \begin{cases} 1 & \text{if } M_n \geq c \\ 0 & \text{o.w.} \end{cases}$$

$\Rightarrow$ The acceptance region is $A(\theta) = \{x : M_n < c\} = \{x : M_n < (1-\alpha)^{\frac{1}{n}}\theta_0\}$.

By the duality thm,

$$S(x) = \{\theta \in \Theta : \theta > M_n(1-\alpha)^{-\frac{1}{n}}\}.$$

$\therefore$ The level $(1-\alpha)$ LCB of θ is $M_n(1-\alpha)^{-\frac{1}{n}}$. //

(c) $P_{\theta_0}(M_n \geq c) = 1-\left(\frac{c}{\theta_0}\right)^n = 1-\alpha \Rightarrow c = \theta_0(\alpha)^{\frac{1}{n}}$.

Then $M_n \geq \theta_0 \alpha^{\frac{1}{n}} \Rightarrow \theta_0 \leq M_n \alpha^{-\frac{1}{n}}$

$\therefore$ The level $(1-\alpha)$ UCB of θ is $M_n \alpha^{-\frac{1}{n}}$.

Further, $\left(M_n(1-\alpha_1)^{-\frac{1}{n}},\ M_n \alpha_2^{-\frac{1}{n}}\right)$ is $(1-\alpha)$ C.I.

where $\alpha_1 + \alpha_2 = \alpha$. //

(d)

The length of interval $\left(M_n (1-\alpha_1)^{-\frac{1}{n}},\ M_n \alpha_2^{-\frac{1}{n}} \right) \underset{\substack{\uparrow \\ \text{let} \\ \alpha_2 = \alpha - \alpha_1}}{=} \left(M_n (1-\alpha_1)^{-\frac{1}{n}},\ M_2 (\alpha-\alpha_1)^{-\frac{1}{n}} \right)$ is

$$d \equiv M_n (\alpha-\alpha_1)^{-\frac{1}{n}} - M_n (1-\alpha_1)^{-\frac{1}{n}} = M_n \left[(\alpha-\alpha_1)^{-\frac{1}{n}} - (1-\alpha_1)^{-\frac{1}{n}} \right] \quad \text{for } \alpha_1 \in [0,1].$$

$$\frac{\partial d}{\partial \alpha_1} = M_n \left[\frac{1}{n} (\alpha-\alpha_1)^{-\frac{1}{n}-1} - \frac{1}{n} (1-\alpha_1)^{-\frac{1}{n}-1} \right] > 0$$

$\Rightarrow$ d is increasing in $\alpha_1 \in [0,1]$.

$\Rightarrow$ The length of interval, d, is minimized at $\alpha_1 = 0$.

$\therefore$ The shortest interval is $\left(M_n,\ M_n \alpha^{-\frac{1}{n}} \right)$. //.

4.7.1.

(a)

Consider two r.v.s. $X \sim \text{Gamma}(r,1)$, $Y \sim \text{Gamma}(s,1)$ and $X \perp Y$.

Then, by Thm B.2.3, $\frac{X}{X+Y} \sim \text{beta}(r,s)$. i.e., $\theta \overset{d}{=} \frac{X}{X+Y}$.

$$\Rightarrow \lambda = \frac{s\theta}{r(1-\theta)} \overset{d}{=} \frac{s \frac{X}{X+Y}}{r \frac{Y}{X+Y}} = \frac{X/2r}{Y/2s}$$

where $X \sim \text{Gamma}(r,1) \overset{d}{=} \chi^2_{2r}$

$Y \sim \text{Gamma}(s,1) \overset{d}{=} \chi^2_{2s}$ and $X \perp Y$

$\therefore$ $\lambda \sim F_{2r,2s}$. //

(b)

$$\pi(\theta|x) \propto f(x|\theta)\pi(\theta) \propto \theta^{x}(1-\theta)^{n-x} \times \theta^{r-1}(1-\theta)^{s-1} = \theta^{x+r-1}(1-\theta)^{n-x+s-1}$$

$\therefore\ \theta|x \sim beta(x+r,\ n-x+s)$.

From the result of (a),

$$Pr\left(F_{2(x+r),2(n-x+s),\frac{\alpha}{2}} \le \frac{(n-x+s)\theta}{(x+r)(1-\theta)} \le F_{2(x+r),2(n-x+s),1-\frac{\alpha}{2}} \,\middle|\, X=x \right) = 1-\alpha.$$

$$\Rightarrow Pr\left(\frac{(x+r)s}{(n-x+s)r} F_{2(x+r),2(n-x+s),\frac{\alpha}{2}} \le \lambda \le \frac{(x+r)s}{(n-x+s)r} F_{2(x+r),2(n-x+s),1-\frac{\alpha}{2}} \,\middle|\, X=x \right) = 1-\alpha$$

Thus, the upper and lower credible bounds for $\lambda|x$ is

$$\left(\frac{(x+r)s}{(n-x+s)r} F_{2(x+r),2(n-x+s),\frac{\alpha}{2}},\ \frac{(x+r)s}{(n-x+s)r} F_{2(x+r),2(n-x+s),1-\frac{\alpha}{2}} \right).$$

And as $\frac{\theta}{1-\theta} = \frac{1}{1-\theta} - 1$,

$$1-\alpha = Pr\left(\frac{x+r}{n-x+s} F_{2(x+r),2(n-x+s),\frac{\alpha}{2}} + 1 \le \frac{1}{1-\theta} \le \frac{x+r}{n-x+s} F_{2(x+r),2(n-x+s),1-\frac{\alpha}{2}} + 1 \,\middle|\, X=x \right)$$

$$= Pr\left(\frac{(x+r)F_{2(x+r),2(n-x+s),\frac{\alpha}{2}}}{(x+r)F_{2(x+r),2(n-x+s),\frac{\alpha}{2}} + (n-x+s)} \le \theta \le \frac{(x+r)F_{2(x+r),2(n-x+r),1-\frac{\alpha}{2}}}{(x+r)F_{2(x+r),2(n-x+r),1-\frac{\alpha}{2}} + (n-x+s)} \,\middle|\, X=x \right)$$

Thus, the upper and lower credible for $\theta|x$ is

$$\left(\frac{(x+r)F_{2(x+r),2(n-x+s),\frac{\alpha}{2}}}{(x+r)F_{2(x+r),2(n-x+s),\frac{\alpha}{2}} + (n-x+s)},\ \frac{(x+r)F_{2(x+r),2(n-x+r),1-\frac{\alpha}{2}}}{(x+r)F_{2(x+r),2(n-x+s),1-\frac{\alpha}{2}} + (n-x+s)} \right)$$

//

4.8.4.

For $Y|\theta \sim \text{Binomial}(m, \theta)$, $\theta \sim \text{beta}(r, s)$ and $x|\theta \sim \text{Binomial}(n, \theta)$,

$$\pi(\theta|x) \propto f(x|\theta)\cdot\pi(\theta) \propto \theta^{x}(1-\theta)^{n-x}\,\theta^{r-1}(1-\theta)^{s-1} = \theta^{x+r-1}(1-\theta)^{n-x+s-1}$$

$\therefore\ \theta|x \sim \text{beta}(x+r,\ n-x+s)$.

Since
$$g(y|x) = \frac{g(y,x)}{g(x)} = \frac{\int_0^1 g(y,x|\theta)\,\pi(\theta)\,d\theta}{\int_0^1 g(x|\theta)\,\pi(\theta)\,d\theta} = \frac{\int_0^1 g(y|\theta)\cdot g(x|\theta)\cdot\pi(\theta)\,d\theta}{\int_0^1 g(x|\theta)\,\pi(\theta)\,d\theta}$$

$$= \int_0^1 g(y|\theta)\,\pi(\theta|x)\,d\theta \quad \text{as} \quad \pi(\theta|x) = \frac{g(x,\theta)}{g(x)} = \frac{g(x|\theta)\cdot\pi(\theta)}{\int_0^1 g(x|\theta)\cdot\pi(\theta)\,d\theta},$$

$$g(y|x) = \int_0^1 g(y|\theta)\,\pi(\theta|x)\,d\theta = \int_0^1 \binom{m}{y}\theta^{y}(1-\theta)^{m-y}\cdot\frac{1}{B(x+r,\,n-x+s)}\cdot\theta^{x+r-1}(1-\theta)^{n-x+s-1}\,d\theta$$

$$= \binom{m}{y}\frac{1}{B(x+r,\,n-x+s)}\int_0^1 \underbrace{\theta^{x+y+r-1}(1-\theta)^{n+m+s-y-x-1}}_{\text{a kernel of beta dist. family}}\,d\theta$$

$$= \binom{m}{y}\frac{B(x+y+r,\ n+m+s-y-x)}{B(x+r,\ n-x+s)}.$$

//