

The Standard Presentation of the Lehmann-Scheffé Theorem

Stat 543 Spring 2005

The standard version of the Lehmann-Scheffé isn't phrased in terms of what happens in an exponential family, but rather makes use of completeness. Recall (from the handout on Bahadur's Theorem) the definition of completeness.

Definition 1 A statistic $T(X)$ is called (boundedly) complete if for any (bounded) real valued function $h(t)$

$$E_{\theta} h(T(X)) = 0 \quad \forall \theta \implies P_{\theta} [h(T(X)) = 0] = 1 \quad \forall \theta$$

However, the only general technique for establishing the completeness of a statistic (at least at the level of Stat 543) is to appeal to the fact that in an exponential family, provided the parameter space (in the natural parameterization $\boldsymbol{\eta}$) under consideration includes an open rectangle in \mathfrak{R}^k , the natural sufficient statistic is complete (it is a "complete sufficient statistic"). Thus the usual version of Lehmann-Scheffé stated and proved below, while more general than the one quoted in class (that is only an exponential family result), in typical practice is really no more useful than the in-class result.

Theorem 2 (The Lehmann-Scheffé Theorem) If $T(X)$ is a complete sufficient statistic and $\delta(X)$ is an unbiased estimator for $\gamma(\theta)$, then with

$$\delta^*(t) = E[\delta(X) | T(X) = t]$$

$\delta^*(T(X))$ is a uniformly minimum variance unbiased estimator of $\gamma(\theta)$.

Further, if $\text{Var}_{\theta} \delta^*(T(X)) < \infty \quad \forall \theta$, $\delta^*(T(X))$ is unique. (If $\delta'(X)$ is another UMVUE of $\gamma(\theta)$, then $P_{\theta}[\delta'(X) = \delta^*(T(X))] = 1 \quad \forall \theta$.)

Proof. By the Rao-Blackwell Theorem,

$$\text{Var}_{\theta} \delta^*(T(X)) \leq \text{Var}_{\theta} \delta(X) \quad \forall \theta$$

If we can conclude that regardless of what unbiased estimator we start with, we get the same $\delta^*(T(X))$, we'll have the first conclusion of the theorem. So suppose that $\delta(X)$ and $\delta'(X)$ are unbiased estimators of $\gamma(\theta)$. Let

$$\begin{aligned} \delta^*(t) &= E[\delta(X) | T(X) = t] \quad \text{and} \\ \delta'^*(t) &= E[\delta'(X) | T(X) = t] \end{aligned}$$

Then

$$\begin{aligned} E_{\theta} (\delta^*(T(X)) - \delta'^*(T(X))) &= E_{\theta} E[\delta(X) | T(X)] - E_{\theta} E[\delta'(X) | T(X)] \\ &= E_{\theta} \delta(X) - E_{\theta} \delta'(X) \\ &= \gamma(\theta) - \gamma(\theta) \\ &= 0 \end{aligned}$$

Completeness of $T(X)$ then implies that

$$P_{\theta} [\delta^*(T(X)) = \delta'^*(T(X))] = 1 \quad \forall \theta$$

and the first part of the theorem is proved.

So consider the second part of the theorem. Suppose that $\delta^*(T(X))$ and $\delta'(X)$ are two UMVUE's of $\gamma(\theta)$, and that

$$\text{Var}_{\theta} \delta^*(T(X)) = \text{Var}_{\theta} \delta'(X) < \infty \quad \forall \theta$$

Note that Rao-Blackwellizing $\delta'(X)$ produces $\delta'^*(T(X))$ which is a UMVUE equal to $\delta^*(T(X))$ with θ probability 1 $\forall \theta$. So

$$\text{Var}_{\theta} \delta'^*(T(X)) = \text{Var}_{\theta} \delta'(X) \quad \forall \theta$$

But for any θ

$$\text{Var}_{\theta} \delta'(X) = \text{Var}_{\theta} E[\delta'(X) | T(X)] + E_{\theta} \text{Var} [\delta'(X) | T(X)]$$

But now

$$\text{Var}_\theta \text{E} [\delta' (X) | T (X)] = \text{Var}_\theta \delta'^* (T (X))$$

(so that $\text{E}_\theta \text{Var} [\delta' (X) | T (X)] = 0$) and

$$\text{Var} [\delta' (X) | T (X)] \geq 0$$

This requires that the non-negative random variable $\text{Var} [\delta' (X) | T (X)]$ must be 0 with θ probability 1,

$$P_\theta [\text{Var} [\delta' (X) | T (X)] = 0] = 1$$

Thus

$$P_\theta [\delta' (X) = \text{E} [\delta' (X) | T (X)]] = 1$$

and since $\text{E} [\delta' (X) | T (X)] = \delta'^* (T (X)) = \delta^* (T (X))$ with θ probability 1, the theorem is proved. ■