Theorem on Minimal Sufficiency
Stat 543 Spring 2005

Theorem 1 *Suppose $f(x|\theta)$ $\forall\theta \in \Theta$ is either a probability density for X on $\Re^k$ or a probability mass function for X, and $T(X)$ is sufficient for θ. If the existence of a positive number $k(x, y)$ such that $f(y|\theta) = k(x, y) f(x|\theta)$ $\forall\theta \in \Theta$ implies that $T(y) = T(x)$, then $T(X)$ is minimal sufficient.*

Proof. Let S be any other sufficient statistic. $\exists$ nonnegative functions $g(s, \theta)$ and $h(x)$ such that

$$f(x|\theta) = g(S(x), \theta) h(x)$$

and we may wolog assume that $h(x) > 0$. (If not, I start over after redefining the observation space by throwing out possible values where $h(x) = 0$. The set of points I thus throw away have probability 0 $\forall\theta \in \Theta$.)

Suppose that $S(y) = S(x)$. For all θ

$$\begin{aligned} f(y|\theta) &= g(S(y), \theta) h(y) \\ &= g(S(x), \theta) h(y) \\ &= g(S(x), \theta) h(x) \left(\frac{h(y)}{h(x)}\right) \end{aligned}$$

So for $k(x, y) = h(y)/h(x)$ we see that $f(y|\theta) = k(x, y) f(x|\theta)$ $\forall\theta \in \Theta$ and thus that $T(y) = T(x)$. That is,

$$S(y) = S(x) \Longrightarrow T(y) = T(x) \qquad (*)$$

We need to show that this implies the existence of a function $q(s)$ such that $T(x) = q(S(x))$.

For each $s \in$Range(S), let x_s be a possible value of X such that

$$S(x_s) = s$$

Define $q(s) = T(x_s)$. This definition is unambiguous because of (*). Then

$$\begin{aligned} q(S(x)) &= T\left(x_{S(x)}\right) \\ &= T(x) \end{aligned}$$

where the last equality follow from the fact that $S(x) = S\left(x_{S(x)}\right)$ and implication (*). ■