Stat 543 Note on Arguing Minimal Sufficiency

I've given you 3 tools for arguing that a statistic is minimal sufficient:

1. showing that for a sufficient $T(\mathbf{X})$, if $\mathbf{x}$ and $\mathbf{y}$ have proportional likelihood functions then $T(\mathbf{x}) = T(\mathbf{y})$

2. showing that a statistic of insterest is equivalent to $R(\mathbf{x}, \boldsymbol{\theta})$ (that Dynkin/Lehmann/Scheffé shows to be minimal sufficient)

3. appealing to an exponential family result

Exactly what is needed to use 3. is presumably obvious (one has to push things into the right form and show that the set of natural parameters $\boldsymbol{\eta}$ of interest contians an open ball). What is needed for 1. and 2. may be less clear. Let me elaborate.

For 1., one can take make the ratio of likelihoods for $\mathbf{x}$ and $\mathbf{y}$, $f(\mathbf{x}|\boldsymbol{\theta})/f(\mathbf{y}|\boldsymbol{\theta})$ and try to argue that this is a constant function of $\boldsymbol{\theta}$ *only if* $T(\mathbf{x}) = T(\mathbf{y})$. In this regard, it is usually easy to argue the converse, namely that $T(\mathbf{x}) = T(\mathbf{y})$ *implies that* $f(\mathbf{x}|\boldsymbol{\theta})/f(\mathbf{y}|\boldsymbol{\theta})$ is a constant function of $\boldsymbol{\theta}$. BUT THIS IS NOT WHAT IS NEEDED. One must make the ratio $f(\mathbf{x}|\boldsymbol{\theta})/f(\mathbf{y}|\boldsymbol{\theta})$ and carefully see necessary conditions under which it is constant in $\boldsymbol{\theta}$. When $\boldsymbol{\theta}$ is other than 1-dimensional, this is not so obvious as we might like it to be (or even as textbooks like C&B in Example 6.2.14 make it out to be). The problem is that just staring at a function of several variables (the entires of $\boldsymbol{\theta}$) it is not REALLY so obvious that sufficient consitions for it to be constant are also necessary! (In the C&B example, sure enough their conditions will make the ratio constant in the parameters, but their assertion that those conditions are NECESSARY for the ratio to be constant in the parameters is NOT obviously true.) So how does one really carefully make such arguments? The only real handle we have on this problem is algebra and freshman calculus of one variable! If $f(\mathbf{x}|\boldsymbol{\theta})/f(\mathbf{y}|\boldsymbol{\theta})$ is constant in $\boldsymbol{\theta}$, it is constant in any single entry of a k-dimensional $\boldsymbol{\theta}$ when the other entries are (set at convenient values and) held fixed. The point here is that I can use algebra and freshman calculus to argue that a function of a single real variable is constant only under some condition on $\mathbf{x}$ and $\mathbf{y}$. Usually, if I do that one coordinate of $\boldsymbol{\theta}$ at a time, I can show that in the end the k conditions that I come up with together require that $T(\mathbf{x}) = T(\mathbf{y})$.

What is needed to use 2. is similar. If I am to show that $T(\mathbf{x})$ is equivalent to $R(\mathbf{x}, \boldsymbol{\theta})$, I need to show that $T(\mathbf{x}) = T(\mathbf{y})$ if and only if $R(\mathbf{x}, \boldsymbol{\theta}) = R(\mathbf{y}, \boldsymbol{\theta})$ (as functions of $\boldsymbol{\theta}$). Usually, showing that $T(\mathbf{x}) = T(\mathbf{y})$ implies that $R(\mathbf{x}, \boldsymbol{\theta}) = R(\mathbf{y}, \boldsymbol{\theta})$ is easy. Showing that $R(\mathbf{x}, \boldsymbol{\theta}) = R(\mathbf{y}, \boldsymbol{\theta})$ implies that $T(\mathbf{x}) = T(\mathbf{y})$ is less obvious. But notice that the structure of what needs doing is very similar to the situation in 1. One assumes that $R(\mathbf{x}, \boldsymbol{\theta}) - R(\mathbf{y}, \boldsymbol{\theta}) = 0$ and needs to conclude that $T(\mathbf{x}) = T(\mathbf{y})$. To do this carefully, one reasons that if the function of $\boldsymbol{\theta}$, $R(\mathbf{x}, \boldsymbol{\theta}) - R(\mathbf{y}, \boldsymbol{\theta})$, is the 0 function, it is the 0 function in any single entry of a k-dimensional $\boldsymbol{\theta}$ when the other entries are (set at convenient values and) held fixed. Using algebra and freshman calculus I can typically argue that a function of a single real variable is 0 only under some condition on $\mathbf{x}$ and $\mathbf{y}$. Typically, if I do that one coordinate of $\boldsymbol{\theta}$ at a time, I can show that in the end the k conditions that I come up with together require that $T(\mathbf{x}) = T(\mathbf{y})$.