<u>Example</u> $X \sim N(\theta, 1)$ $\theta \sim N(0, \gamma^2)$ a priori

$\theta | X = x$ is $N\left(\frac{\gamma^2}{\gamma^2+1} x, \frac{\gamma^2}{\gamma^2+1}\right)$

Suppose I want to test $H_0: |\theta| \le .3$ vs $H_1: |\theta| > .3$

$\left(\Theta_0 = [-.3, .3],\ \Theta_1 = (-\infty, -.3) \cup (.3, \infty)\right)$

The posterior probability of $\theta \in \Theta_0$ is

$$P[\theta \in \Theta_0 | X = x] = \Phi\left(\frac{.3 - \frac{\gamma^2}{\gamma^2+1} x}{\sqrt{\frac{\gamma^2}{\gamma^2+1}}}\right) - \Phi\left(\frac{-.3 - \frac{\gamma^2}{\gamma^2+1} x}{\sqrt{\frac{\gamma^2}{\gamma^2+1}}}\right)$$

So the Bayes optimal test under 0-1 loss is the indicator that the above is less than .5

<u>Example</u> Simple discrete one with finite parameter space

Suppose for $\theta \in \{1, 2, 3, 4\}$ pmf's $f(x|\theta)$ are as below

θ \ x	1	2	3	4	5	6
1	.05	.1	.025	.3	.375	.15
2	.2	.4	.1	.1	.15	.05
3	.1	.2	.05	.2	.35	.1
4	.05	.1	0	.4	.25	.2

Suppose I use a prior specified by pmf

θ	1	2	3	4
$g(\theta)$	.4	.2	.2	.2

		x: 1	2	3	4	5	6
	1	.02	.04	.01	.12	.15	.06
θ	2	.04	.08	.02	.02	.03	.01
	3	.02	.04	.01	.04	.07	.02
	4	.01	.02	0	.08	.05	.04

So if, for example, I want a 0-1 loss Bayes test of $H_0: \theta = 1$ or 3 vs $H_a: \theta = 2$ or 4 I sum the rows above for 1+3 and the rows for 2+4 and compare in order to identify an optimal rule

	1	2	3	4	5	6
(H_0) θ 1 and 3 sum	.04	.08	.02	.16	.22	.08
(H_a) θ 2 and 4 sum	.05	.10	.02	.10	.08	.05

So a Bayes test has

$$\phi(x) = \begin{cases} 1 & \text{if } x = 1,2 \\ 0 & \text{if } x = 4,5,6 \end{cases}$$

($\phi(3)$ can be anything 0 to 1 inclusive)

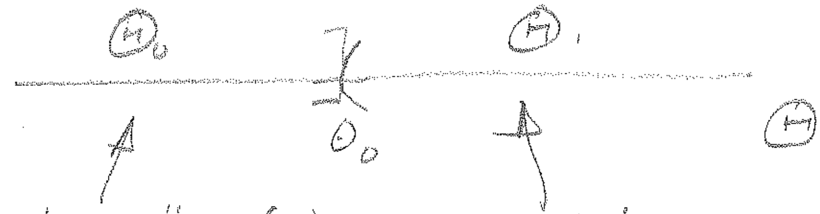
day 23

What about non-Bayes optimality theory for other than simple vs simple testing?

There is some optimality theory for some "one-sided" one-parameter problems -

What might one want?

$$H_0 : \theta \le \theta_0 \quad \text{vs} \quad H_1 : \theta > \theta_0$$



want small $\pi_\phi(\theta)$ there   want large $\pi_\phi(\theta)$ there

Since I'm out of the "simple null hypothesis" world, I need a more general definition of "size" of a test

<u>Def</u> The size of a test $\phi(x)$ is

$$\text{really } \overset{\sup}{\longrightarrow} \ \underset{\theta \in \Theta_0}{\text{"max"}} \ E_\theta\, \phi(X) = \underset{\theta \in \Theta_0}{\text{"max"}} \ \pi_\phi(\theta)$$

It would be "ideal" if we could look around in the set of ϕ's of size $\le \alpha$ and find one that has the largest possible power

<u>Def</u> A test $\phi(x)$ is UMP (<u>u</u>niformly (in θ across Θ_1) <u>m</u>ost <u>p</u>owerful) of size α provided

1) $\pi_\phi(\theta) \le \alpha \quad \forall\ \theta \in \Theta_0$

and 2) for any other test ϕ' of size $\le \alpha$

$$\pi_\phi(\theta) \ge \pi_{\phi'}(\theta) \quad \forall\ \theta \in \Theta_1$$

How to find these in some limited contexts is illustrated in the <u>archetypal</u> <u>example</u>

Archetypal Example (of development of a UMP test)

$X \sim N(\theta, 1)$

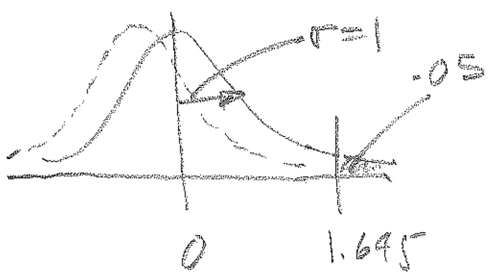
$H_0: \theta \le 0$ vs $H_1: \theta > 0$

Suppose that I'm interested in size $\alpha = .05$ tests - a perfectly plausible one is

$$\phi(x) = \begin{cases} 1 & \text{if } x > 1.645 \\ 0 & \quad < \end{cases}$$

This is, in fact, UMP size $\alpha = .05$ for these hypotheses - how to argue this?

Step 1 Note that ϕ is of size $\alpha = .05$



$\pi_\phi(\theta) = 1 - \Phi(1.645 - \theta)$

for $\theta \le 0 \Rightarrow \le 1 - \Phi(1.645)$

$= .05$

i.e. $\sup_{\theta \le 0} \pi_\phi(\theta) = \pi_\phi(0) = .05$

Step 2 Note that for $\theta_1 > 0$, ϕ is a MP size $\alpha = .05$ test of $H_0: \mu = 0$ vs $H_1: \mu = \theta_1$ Why?

$$\frac{f(x \mid \theta_1)}{f(x \mid 0)} = \frac{\exp -\frac{1}{2}(x^2 - 2\theta_1 x + \theta_1^2)}{\exp -\frac{1}{2}x^2}$$

$$= \exp(\theta_1 x) \exp(-\tfrac{1}{2}\theta_1^2) \nearrow \text{ in } x$$

since $\theta_1 > 0$

Hence ϕ is of the right form to apply the NP Lemma

Step 3 Note that step 2 implies that ϕ is UMP size $\alpha = .05$ for $H_0: \theta = 0$ vs $H_1: \theta > 0$

Why? NP says that any other test ϕ' with

$$\pi_{\phi'}(0) \le \pi_{\phi}(0) = .05$$

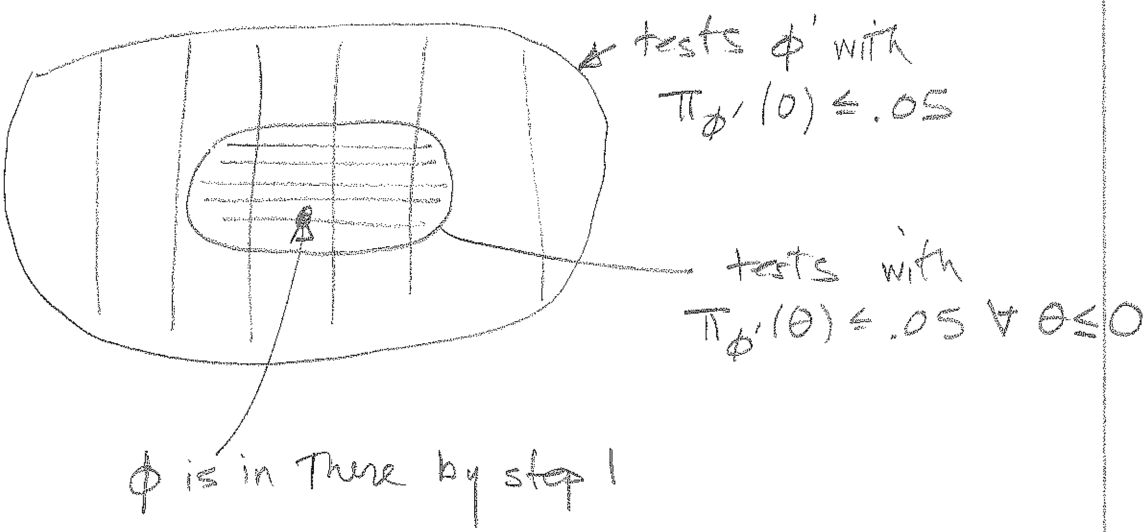
has $\pi_{\phi'}(\theta_1) \le \pi_{\phi}(\theta_1)$ (for each $\theta_1 > 0$!)

Step 4 Recognize that the set of tests ϕ' with

$$\pi_{\phi'}(\theta) \le .05 \quad \forall \theta \le 0$$

is a subcollection of the tests ϕ' with

$$\pi_{\phi'}(0) \le .05$$



Step 5 Since from step 3, if ϕ' is in

it must be the case that

$$\pi_{\phi'}(\theta) \le \pi_{\phi}(\theta) \quad \forall \theta > 0$$

it is also true that if ϕ' is in

$$\pi_{\phi'}(\theta) \le \pi_{\phi}(\theta) \quad \forall \theta > 0$$

Moden: the toughest guy in town is also the toughest guy on his block

But this is exactly that ϕ is UMP level $\alpha = .05$ for $H_0: \theta \le 0$ vs $H_a: \theta > 0$

The preceding is a very important example — the following shows how far it is possible to generalize the argument — the very key idea is that for any $\theta_0 < \theta_1$

$$R(x) = \frac{f(x|\theta_1)}{f(x|\theta_0)}$$

this (statistic) doesn't depend on θ_1 $(> \theta_0)$

is increasing in x, so that rejecting H_0 for large x substitutes for rejecting for large $R(x)$ for all $\theta_0 < \theta_1$

Consider $\Theta \subset \mathbb{R}^1$ and hypotheses

A $\quad H_0: \theta \le \theta_0$, $H_1: \theta > \theta_0$ $\quad$ or $\quad$ B $\quad H_0: \theta \ge \theta_0$, $H_1: \theta < \theta_0$

where X has pdf or pmf $f(x|\theta)$

Def The family of distributions specified by $f(x|\theta)$ has monotone likelihood ratio provided for $\theta_0 < \theta_1$ both belonging to Θ

$$\frac{f(x|\theta_1)}{f(x|\theta_0)}$$

is non-decreasing in some $T(x)$ on

$$\{x \mid f(x|\theta_1) + f(x|\theta_0) > 0\}$$

day 29

Example $N(\mu, 1)$ for $\mu_1 < \mu_2$

$$\frac{f(x|\mu_2)}{f(x|\mu_1)} = \exp -\frac{1}{2}\left[(x-\mu_2)^2 - (x-\mu_1)^2\right]$$

$$= \exp\left[-\frac{1}{2}(\mu_2^2 - \mu_1^2) + x(\mu_2 - \mu_1)\right]$$

and this is clearly monotone non-decreasing in x, i.e. we have MLR in $T(X) = X$

Example Binomial (n, p) family for $p_1 < p_2$

$$\frac{f(x|p_2)}{f(x|p_1)} = \left(\frac{p_2}{p_1}\right)^x \left(\frac{1-p_2}{1-p_1}\right)^{n-x}$$

$$= \left[\frac{p_2/(1-p_2)}{p_1/(1-p_1)}\right]^x \left(\frac{1-p_2}{1-p_1}\right)^n$$

and this is monotone non-decreasing in x, i.e. we have MLR in X

Example single parameter exponential family in natural parameterization — for a single observation

B+D form

$$f(x|\eta) = h(x)\exp(\eta T(x) - A(\eta))$$

$X_1, X_2, \ldots, X_n$ iid $f(x|\eta)$ has joint pdf

$$f(x|\eta) = \prod_{i=1}^{n} h(x_i)\exp(\eta \sum T(x_i - nA(\eta)))$$

So the likelihood ratio for $\eta_1 < \eta_2$ is

$$\frac{f(x|\eta_2)}{f(x|\eta_1)} = \exp\left[(\eta_2 - \eta_1)\sum T(x_i)\right]\exp\left[-n\left(A(\eta_2) - A(\eta_1)\right)\right]$$

and this is clearly monotone non decreasing in $\sum T(x_i)$, i.e. there is MLR in $\sum T(X_i)$

Example single parameter exponential family with

$$f(x|\theta) = h(x)\exp(\eta(\theta)T(x) - A(\eta(\theta)))$$

if $\eta(\theta)$ is increasing in θ we have, again, MLR in $T(x)$

Note BTW that if

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} \text{ is non-increasing in } T(x)$$

The family has MLR in $-T(X)$

There are a number of convenient mathematical properties that follow from MLR – for one, if one assumes that $\theta \sim g(\theta)$ and there is MLR in X or $-X$, $E[\theta | X = x]$ is monotone in x

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