Stat 543 Exam 1

February 21, 2005 **Prof. Vardeman**

WRITE ALL YOUR ANSWERS ON THIS EXAM.

1. This question concerns observations from a (two-parameter exponential) distribution with marginal pdf on $\Re$

$$f(x \mid \beta, \tau) = I[x \geq \tau] \frac{1}{\beta} \exp\left(-\left(\frac{x-\tau}{\beta} \right) \right)$$

for $\beta > 0$ and $\tau \in \Re$. (If X has this distribution then $X = Y + \tau$ where Y has the usual exponential distribution on $(0, \infty)$ with mean β.)

Suppose that $X_1, X_2, \ldots, X_n$ are iid with pdf $f(x \mid \beta, \tau)$.

a) Identify a two-dimensional sufficient statistic for the parameter vector (β, τ) and carefully argue that it is sufficient.

b) Find a "method of moments" estimator of the parameter vector (β, τ) based on theoretical and sample means and standard deviations.

c) For any $(\beta,\tau)\in(0,\infty)\times\Re$ the probability is 1 that the likelihood has a maximum. Identify the maximum likelihood estimator, and carefully argue that it does indeed maximize the likelihood.

e) Consider Bayes inference under the "improper prior distribution" on $(0,\infty)\times\Re$ with "pdf" $g(\beta,\tau)=1$. It is possible to show (don't try to do so here) that for likelihood function $L(\beta,\tau)$,

$$D(\mathbf{x})\equiv\iint L(\beta,\tau)\,d\tau\,d\beta=\frac{\Gamma(n-2)}{n\left(\sum_{i=1}^{n}(x_i-\min x_i)\right)^{n-2}}<\infty .$$

Use this fact and write out a formula (an iterated integral complete with limits of integration) for the (squared error loss) Bayes estimator of τ. (DO NOT take time to do so, but one should be able to begin with only your expression and use calculus to arrive at an explicit formula for the estimator.)

2. Consider for $\eta > 0$ the family of distributions on $[0,1]$ with pdfs

$$f(x \mid \eta) = C(\eta)\exp\left(-\frac{1}{2}\eta x^2\right) \text{ where } C(\eta) = \frac{\sqrt{\eta}}{\sqrt{2\pi}\left[\Phi\left(\sqrt{\eta}\right) - .5\right]}$$

(In this problem, Φ and ϕ are respectively the standard normal cdf and pdf.) Suppose that $X_1, X_2, \ldots, X_n$ are iid with pdf $f(x \mid \eta)$.

a) Identify a minimal sufficient statistic here and argue carefully that it is indeed minimal sufficient.

b) As it turns out

$$m_1(\eta) \equiv \mathrm{E}_\eta X_1 = \frac{C(\eta)}{\eta}\left(1 - \exp\left(-\frac{\eta}{2}\right)\right) \text{ and } m_2(\eta) \equiv \mathrm{E}_\eta X_1^2 = -\frac{1}{2\eta} + \frac{\phi\left(\sqrt{\eta}\right)}{2\sqrt{\eta}\left[\Phi\left(\sqrt{\eta}\right) - .5\right]}$$

One could parameterize this family of distributions by the marginal mean, $\mu = \mathrm{E}X_1$, by writing the marginal pdf on $[0,1]$ as $h(x \mid \mu) = f\left(x \mid m_1^{-1}(\mu)\right)$ (for m_1^{-1} the inverse function for m_1). Give a formula for the MLE of μ based on $X_1, X_2, \ldots, X_n$. (In this formula, you may use any of the functions $m_1(\bullet), m_2(\bullet), C(\bullet)$ and their inverses without writing them out.)

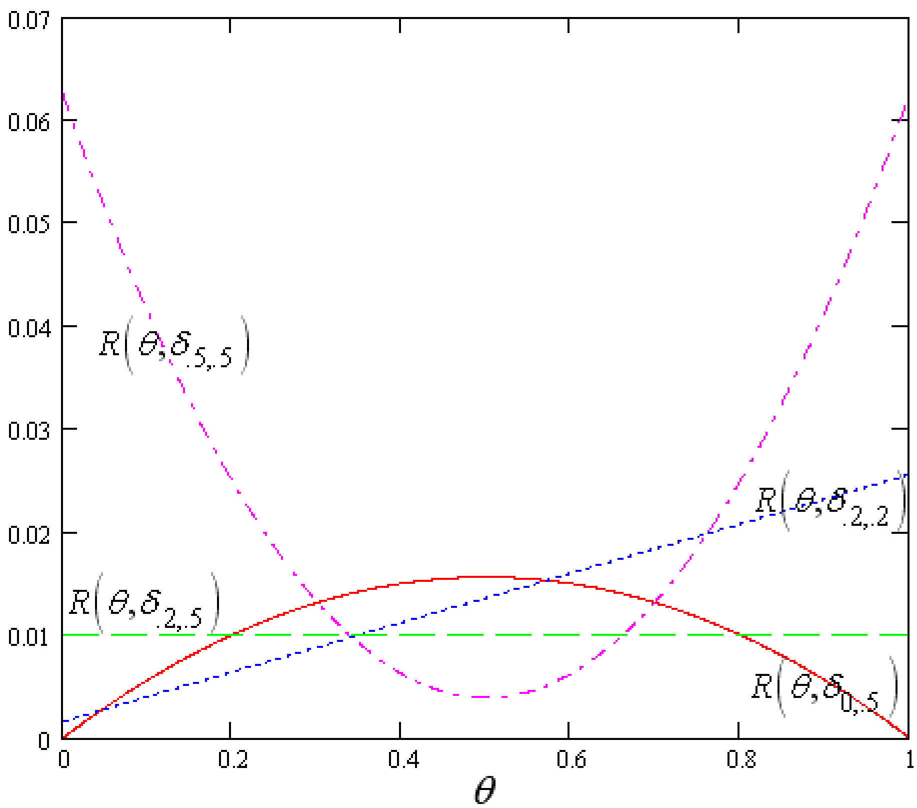
3. In a decision problem, $\theta \in [0,1]$ and $a \in [0,1]$. The loss function is $L(\theta, a) = (\theta - a)^2$ and the observable is $X \sim \text{Bi}(16, \theta)$. Under consideration are several decision functions of the form

$$\delta_{\alpha,c}(x) = \alpha c + (1-\alpha)\frac{x}{16}$$

for constants $\alpha \in [0,1]$ and $c \in [0,1]$.

a) Find an explicit expression for the risk function corresponding to $\delta_{\alpha,c}(x)$. (You may quit simplifying when it is clear that your risk is quadratic in θ.)

The figure below shows plots of the risk functions for 4 of these decision functions, $\delta_{0,.5}, \delta_{.2,.2}, \delta_{.2,.5}$ and $\delta_{.5,.5}$.



b) Which decision function is most attractive from a "minimax" point of view? (Circle one.)

$\delta_{0,.5}$ $\delta_{.2,.2}$ $\delta_{.2,.5}$ $\delta_{.5,.5}$

c) Which decision function is most attractive to a Bayesian with a prior that is

i) uniform on the interval $(0,.2)$ (circle one)

$\delta_{0,.5}$ $\delta_{.2,.2}$ $\delta_{.2,.5}$ $\delta_{.5,.5}$

ii) uniform on the interval $(.4,.6)$ (circle one)

$\delta_{0,.5}$ $\delta_{.2,.2}$ $\delta_{.2,.5}$ $\delta_{.5,.5}$

iii) uniform on the pair of values $\{.1,.9\}$ (circle one)

$\delta_{0,.5}$ $\delta_{.2,.2}$ $\delta_{.2,.5}$ $\delta_{.5,.5}$

4. An observable X has a distribution specified by one of the pmfs in the table below.

		x					
		1	2	3	4	5	6
θ	1	.05	.10	.025	.30	.375	.15
	2	.20	.40	.10	.10	.15	.05
	3	.10	.20	.05	.20	.35	.10
	4	.05	.10	0	.40	.25	.20

a) Specify values $T(x)$ so that $T(X)$ is a minimal sufficient statistic when $\Theta=\{1,2,3,4\}$.

$T(1)=$ ____ $T(2)=$ ____ $T(3)=$ ____ $T(4)=$ ____ $T(5)=$ ____ $T(6)=$ ____

b) Is your statistic from a) sufficient when $\Theta=\{1,2,3\}$? Explain.

c) Is your statistic from a) minimal sufficient when $\Theta=\{1,2,3\}$? If so, say why. If not, give values $S(x)$ so that $S(X)$ (not equivalent to $T(X)$) is minimal sufficient.

5. A family of discrete distributions has pmfs

$$f(y \mid \lambda) = \frac{\exp(-\lambda)\lambda^{y}}{y!}\left(1 + \frac{\lambda}{y+1}\right) \text{ for } y = 0, 2, 4, 6\ldots$$

For $Y_1, Y_2, \ldots, Y_n$ iid with pmf $f(y \mid \lambda)$, consider maximum likelihood estimation of λ. Completely describe an E-M algorithm that can (presumably) be used to find a maximizer of the likelihood.