Stat 543 Exam 1
Spring 2016

I have neither given nor received unauthorized assistance on this exam.

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Name Signed **Date**

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Name Printed

This Exam consists of 9 questions that will be scored at 10 points apiece (making 90 points possible).

There is also on the last page of the Exam an "Extra Credit" question that will be scored out of 10 points. Any Extra Credit obtained will be recorded and used at the end of the course at Vardeman's discretion in deciding borderline grades. DO NOT spend time on this question until you are done with the entirety of the regular exam.

1. Consider two distributions for a bivariate random vector $\mathbf{X} = (X_1, X_2)$ taking values in the unit square $[0,1]^2$ with (joint) pdfs

$$f(\mathbf{x} \mid 0) = I\left[\mathbf{x} \in [0,1]^2\right] \text{ and } f(\mathbf{x} \mid 1) = (x_1 + x_2) I\left[\mathbf{x} \in [0,1]^2\right]$$

a) Identify a minimal sufficient statistic for the two-class model $\mathcal{P} = \{P_0, P_1\}$ corresponding to these two pmfs.

b) Set up completely (including limits of integration), but do not try to evaluate, a double integral giving the K-L information regarding $f(\mathbf{x} \mid 1)$ relative to $f(\mathbf{x} \mid 0)$.

c) For a uniform prior distribution (on the two-point parameter space $\Theta = \{0,1\}$), i.e. one with $g(0) = g(1) = \frac{1}{2}$, what is the posterior probability of $\theta = 1$ given the observation $\mathbf{X} = (x_1, x_2)$? (Give a function $g(1 | \mathbf{x}) : \Re^2 \to [0,1]$.)

d) For a decision problem with $\mathcal{A} = \Theta = \{0,1\}$ and 0-1 loss ($L(\theta, a) = I[a \neq \theta]$) and the uniform prior of part **c)** what is a Bayes optimal decision function, $d : [0,1]^2 \to \{0,1\}$? (Identify the set of $\mathbf{x} \in [0,1]^2$ for which one should take action $a = 1$.)

e) For your decision function, d, from part **d)**, evaluate the two values of the risk function $R(\theta,d)$, namely $R(0,d)$ and $R(1,d)$. (If you could not do part **d)** you may use the incorrect decision function $d(\mathbf{x}) = I[x_1 < x_2]$.)

2. Consider a class of discrete distributions on the small sample space $\mathcal{X} = \{0,1,2\}$ with pmfs

$$f(x \mid \eta) \propto \exp(\eta x^2)$$

for parameter space $\Gamma = \Re$. Suppose that $X_1, X_2, \ldots, X_n$ are iid according to pmf $f(x \mid \eta)$.

a) Identify a minimal sufficient statistic $T(\mathbf{X})$ in this context and argue very carefully that it is indeed minimal sufficient (applying whatever results from class or B&D are helpful).

b) Consider method of moments estimation of η based on $\widehat{\mu_1}_n = \overline{x}$. Find the estimating equation that a MOM estimator $\hat{\eta}_n^{\text{MOM}}$ must solve here.

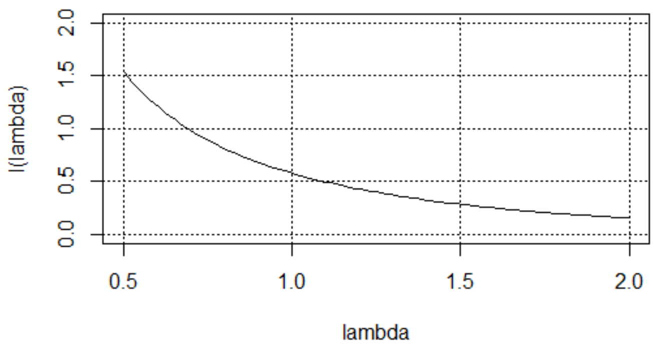
c) Consider maximum likelihood estimation of η. Find an estimating equation that the Maximum Likelihood estimator $\hat{\eta}_n^{\text{MLE}}$ must solve here.

d) What basis is there for Vardeman's expectation that the MLE from **c)** will have better theoretical properties than the MOM estimator from **b)**?

3. Consider a situation in which $X \sim \text{Poisson}(\lambda)$ for $\lambda \in \Re$ but one observes only the (Bernoulli) variable $Y = I[X > 0]$.

a) Find the Fisher Information in Y about λ at the parameter value λ_0.

b) Below is a plot of the correct answer to **a)**. Suppose that in fact $\lambda = 1.5$. How large must n be in order for a sample $Y_1, Y_2, \ldots, Y_n$ (derived from independent $\text{Poisson}(\lambda)$ variables $X_1, \ldots, X_n$ as indicated above) to be as informative about λ as a single $X \sim \text{Poisson}(\lambda)$?



4. Extra Credit – (Weighted Squared Error Loss) Suppose that in a Bayes statistical model some real-valued parametric function $\gamma(\boldsymbol{\theta})$ is of interest and will be estimated under weighted squared error loss

$$L(\boldsymbol{\theta}, a) = w(\boldsymbol{\theta})(\gamma(\boldsymbol{\theta}) - a)^2$$

for a known function $w(\boldsymbol{\theta}) \geq 0$. By considering the posterior distribution for observed data $X = x$, describe an optimal $d(x)$ in terms of expectations derived from this conditional distribution.