

**Stat 543 Exam 2  
Spring 2016**

**I have neither given nor received unauthorized assistance on this exam.**

KEY

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**Name Signed**

**Date**

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**Name Printed**

This Exam consists of 11 questions. Do at least 10 of the 11 parts of the main exam. I will score your best 10 answers at 10 points apiece (making 100 points possible).

There is also on the last page of the Exam an "Extra Credit" question that will be scored out of 10 points. Any Extra Credit obtained will be recorded and used at the end of the course at Vardeman's discretion in deciding borderline grades. DO NOT spend time on this question until you are done with the entirety of the regular exam.

1. Below are three pmfs for  $X$ ,  $f(x|1)$ ,  $f(x|2)$ , and  $f(x|3)$ . Use them in the rest of this question.

		x						
		1	2	3	4	5	6	7
$.3f(x 3)$ →	3	.20 .06	.05 .015	.15 .045	.15 .045	.10 .03	.05 .015	.30 .09
$.3f(x 2)$ →	2	.10 .03	.05 .015	.25 .075	.05 .015	.20 .06	.15 .045	.20 .06
$.4f(x 1)$ →	1	.25 .1	.15 .06	.05 .02	.25 .1	.15 .06	.05 .02	.10 .04
$R(x)$		.4	.33	.5	.2	1.33	3	2

biggest sum (part c)

a) For which  $\alpha$  are there non-randomized most powerful size  $\alpha$  tests of  $H_0: \theta=1$  vs  $H_1: \theta=2$ ?

Possible  $\alpha$  are 0, .05, (.05+.05)=.1, (.05+.05+.1)=.2, (.05+.05+.1+.15)=.35, (.05+.05+.1+.15+.25)=.6, (.05+.05+.1+.15+.25+.15)=.75, 1

b) Identify a most powerful size  $\alpha = .15$  test of  $H_0: \theta=1$  vs  $H_1: \theta=2$ .

$$\phi(x) = \begin{cases} 1 & \text{if } x=3,6 \\ .5 & \text{if } x=7 \\ 0 & \text{if } x=1,2,4,5 \end{cases}$$

$$E\phi(X) = 1(.05+.05) + \frac{1}{2}(.10) = .15 \text{ as desired}$$

c) Find a 0-1 loss Bayes test of  $H_0: \theta = 1$  vs  $H_1: \theta = 2$  or  $3$  for a prior distribution with  $g(1) = .4, g(2) = .3$ , and  $g(3) = .3$ . (Give all 7 values of  $\phi(x)$ .)

$$\phi(x) = \begin{cases} 1 & \text{if } x = 3, 5, 6, 7 \\ 0 & \text{if } x = 1, 2, 4 \end{cases}$$

See the computations on the table. Compare  $.4f(x|1)$  to  $.3f(x|2) + .3f(x|3)$

2. In this problem we'll use the  $\text{Exp}(\lambda)$  distribution with pdf  $f(x|\lambda) = \lambda \exp(-\lambda x) I[x > 0]$ . You may use without proof the facts that

- if  $X \sim \text{Exp}(\lambda)$  and  $t > 0$  then  $P[X > t] = \exp(-\lambda t)$ ,
- if  $X_1 \sim \text{Exp}(\lambda)$  independent of  $X_2 \sim \text{Exp}(\lambda_2)$  then  $Y \equiv \min(X_1, X_2) \sim \text{Exp}(\lambda_1 + \lambda_2)$

In a so-called "competing risks" context, an individual or item has a lifetime  $Z = \min(U, V)$  where  $U$  and  $V$  are positive times to failure/death from two different causes.

a) For  $i = 1, \dots, n$  model  $U_i \sim \text{Exp}(\lambda_1)$  and  $V_i \sim \text{Exp}(\lambda_2)$  with all  $U$ 's and  $V$ 's independent. Suppose that what is observed are the iid pairs  $\mathbf{W}_i = (Z_i, I[Z_i = U_i])$ . (Note that  $I[Z_i = U_i] = 1$  means that what is observed is the value of  $U_i$  and the fact that  $U_i < V_i$ .) Give likelihood terms  $f(\mathbf{w} | \lambda_1, \lambda_2)$  for observed  $\mathbf{w}_i = (z_i, 1)$  and  $\mathbf{w}_i = (z_i, 0)$ .

$$f((z_i, 1) | \lambda_1, \lambda_2):$$

$$\lambda_1 \exp(-\lambda_1 z_i) \exp(-\lambda_2 z_i)$$

$$f((z_i, 0) | \lambda_1, \lambda_2):$$

$$\exp(-\lambda_1 z_i) \lambda_2 \exp(-\lambda_2 z_i)$$

(Note, e.g. that  $P[U < V] = \int_0^\infty \int_u^\infty \lambda_1 \exp(-\lambda_1 u) \lambda_2 \exp(-\lambda_2 v) dv du = \frac{\lambda_1}{\lambda_1 + \lambda_2}$ )

$$\text{So } P[U < t | U < V] = \int_0^t \exp(-\lambda_2 u) \lambda_1 \exp(-\lambda_1 u) du / \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right)$$

$$\frac{d}{dt}(\text{above}) = \left( \lambda_1 \exp(-(\lambda_1 + \lambda_2)t) \right) / \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right) \text{ and so}$$

the likelihood term for the  $w_2 = 1$  case is  $\left( \text{this times } \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right) \right)$

b) Sometimes, a cause of failure may not be recorded and thus only  $Z_i$  (and not  $w_i$ ) is known. Suppose that information on  $n = 5$  individuals/items is  $w_1 = (3,1), w_2 = (7,1), Z_3 = 2, w_4 = (3,0)$ , and  $w_5 = (1,0)$ . Suppose further that a Bayesian uses a prior for  $(\lambda_1, \lambda_2)$  that is one of independence with both  $\lambda_1$  and  $\lambda_2$  a priori  $\text{Exp}(1)$  distributed. Carefully describe a Gibbs sampling algorithm for generating triples  $((\lambda_1)_j^*, (\lambda_2)_j^*, (w_{3,2})_j^*)$  (iterates for the 2 rates and the unobserved indicator). If it is possible to name a distribution from which a given update must be sampled, do so. At a minimum, give a form for each univariate update distribution up to a multiplicative constant.

If one has available  $w_{3,2} = \mathbb{I}[z_3 = u_3]$  the likelihood is

$$\begin{aligned} \prod_{i=1}^5 f(z_i, w_i | \lambda_1, \lambda_2) &= \lambda_1 \exp(-3\lambda_1) \exp(-3\lambda_2) \times \lambda_1 \exp(-7\lambda_1) \exp(-7\lambda_2) \\ &\quad \times \lambda_1^{w_{3,2}} \lambda_2^{1-w_{3,2}} \exp(-2\lambda_1) \exp(-2\lambda_2) \\ &\quad \times \lambda_2 \exp(-3\lambda_1) \exp(-3\lambda_2) \times \lambda_2 \exp(-1\lambda_1) \exp(-1\lambda_2) \\ &= \lambda_1^{2+w_{3,2}} \lambda_2^{2+(1-w_{3,2})} \exp(-16\lambda_1) \exp(-16\lambda_2) \end{aligned}$$

and  $g(\lambda_1, \lambda_2) = \exp(-\lambda_1 - \lambda_2)$ , so the posterior is proportional to

$$h(\lambda_1, \lambda_2, w) = \lambda_1^{2+w} \lambda_2^{3-w} \exp(-17\lambda_1) \exp(-17\lambda_2)$$

So Gibbs updates (say in the order  $\lambda_1, \lambda_2, w$ ) are

$$(\lambda_1)_{j+1}^* \sim \Gamma(3 + (w_{3,2})_j^*, 17)$$

$$(\lambda_2)_{j+1}^* \sim \Gamma(4 - (w_{3,2})_j^*, 17)$$

and  $(w_{3,2})_{j+1}^*$  is Ber where the "success probability" "p" has

$$\frac{p}{1-p} = \frac{(\lambda_1)_{j+1}^*}{(\lambda_2)_{j+1}^*} \quad \text{i.e.} \quad p = \frac{(\lambda_1)_{j+1}^*}{(\lambda_1)_{j+1}^* + (\lambda_2)_{j+1}^*}$$

$$\text{i.e.} \quad (w_{3,2})_{j+1}^* \sim \text{Ber} \left( \frac{(\lambda_1)_{j+1}^*}{(\lambda_1)_{j+1}^* + (\lambda_2)_{j+1}^*} \right)$$

c) Completely describe an EM algorithm that can be used to find an MLE of  $(\lambda_1, \lambda_2)$  based on the data used in part b). (It is not really necessary to resort to EM here, as the calculus problem is fairly easy. But for purposes of the exam, write out the EM algorithm.)

The likelihood based on all  $S$   $w_i$ 's is as before

$$\lambda_1^{2+W_{3,2}} \lambda_2^{3-W_{3,2}} \exp(-16\lambda_1) \exp(-16\lambda_2)$$

with logarithm

$$(2+W_{3,2}) \log \lambda_1 + (3-W_{3,2}) \log \lambda_2 - 16\lambda_1 - 16\lambda_2$$

For fixed  $\lambda_1, \lambda_2$

$$E_{\lambda_1, \lambda_2}(W_{3,2}) = P_{\lambda_1, \lambda_2}[U < V] = \int_0^\infty e^{-\lambda_2 u} \lambda_1 e^{-\lambda_1 u} du$$

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

So, the expectation step is to replace  $w_{3,2}$  above with

$$\left(2 + \frac{\lambda_{1j}}{\lambda_{1j} + \lambda_{2j}}\right) \log \lambda_1 + \left(3 - \frac{\lambda_{1j}}{\lambda_{1j} + \lambda_{2j}}\right) \log \lambda_2 - 16\lambda_1 - 16\lambda_2$$

The maximization step can be accomplished by taking partials and setting equal to 0 - This is

$$\frac{1}{\lambda_1} \left(2 + \frac{\lambda_{1j}}{\lambda_{1j} + \lambda_{2j}}\right) - 16 = 0 \quad \text{i.e.} \quad \lambda_{1,j+1} = \frac{\left(2 + \frac{\lambda_{1j}}{\lambda_{1j} + \lambda_{2j}}\right)}{16}$$

$$\text{and} \quad \frac{1}{\lambda_2} \left(3 - \frac{\lambda_{1j}}{\lambda_{1j} + \lambda_{2j}}\right) - 16 = 0 \quad \text{i.e.} \quad \lambda_{2,j+1} = \frac{\left(3 - \frac{\lambda_{1j}}{\lambda_{1j} + \lambda_{2j}}\right)}{16}$$

3. Suppose that  $X_1, X_2, \dots, X_n$  are iid  $\text{Ber}(p)$ . Let  $S_m = \sum_{i=1}^m X_i$ . A statistician expecting to have only  $n-1$  observations  $X_i$  available for inference develops an estimator  $\delta(S_{n-1})$  for  $p$  under SEL. (This estimator may well be a biased estimator.) In fact,  $n$  observations will be available. Find another estimator of  $p$ , say  $\delta^*(S_n)$ , that you are sure will have smaller  $\text{MSE}_p$  than  $\delta(S_{n-1})$  no matter what is the value of  $p \in (0,1)$ .

4. Suppose that  $X_1, X_2, \dots, X_n$  are iid with marginal pdf  $f(x|\alpha) = \alpha x^{\alpha-1} I[0 < x < 1]$ .

a) Find a lower bound for the variance of any unbiased estimator  $\delta(\mathbf{X})$  of  $\gamma(\alpha) = \sin \alpha$  (based on the vector of  $n$  observations).

b) Do you expect there to exist an unbiased estimator of  $\gamma(\alpha) = \sin \alpha$  achieving your bound from a)? Explain!

5. Suppose that  $X_1, X_2, \dots, X_n$  are iid  $\text{Bi}(m, p)$  and (perhaps for "acceptance sampling" purposes)

$$\gamma(p) = P_p[X > 0] = 1 - (1 - p)^m$$

is of interest. **Find** an UMVUE for this quantity and **say** why you know your estimator is UMVU.

(Hint: You may find it useful to think of the  $X_i$ 's as  $\sum_{j=1}^m Y_{i,j}$  for  $m \cdot n$  independent variables  $Y_{i,j}$  each  $\text{Ber}(p)$  .)





6. Argue carefully that you could use iid double exponential observations (i.e. ones with marginal pdf  $\frac{1}{2}\exp(-|x|)$  on  $\mathfrak{R}$ ) to generate a standard normal random variable via the rejection algorithm, but that you could NOT use iid standard normal random variables to generate a double exponential random variable via the rejection algorithm.

7. (EXTRA CREDIT ONLY) Consider the two marginal pdfs

$$f(x|0) = I[0 < x < 1] \quad \text{and} \quad f(x|1) = \frac{2^x}{\ln(2)} I[0 < x < 1]$$

and iid observations  $X_1, X_2, \dots$  from one of these distributions (specified by  $f(x|\theta)$ ). Argue carefully that there exists a non-randomized UMP test of  $H_0 : \theta = 0$  vs  $H_1 : \theta = 1$  for any size  $\alpha \in (0, 1)$  based on  $X_1, \dots, X_n$ . Give an explicit large  $n$  approximate form for such a test for  $\alpha = .05$ .