Stat 543 Exam 2

April 4, 2005 **Prof. Vardeman**

WRITE ALL YOUR ANSWERS ON THIS EXAM!!!

1. It is a Stat 542 probability fact that if $X_1, X_2, \ldots, X_n$ are iid Poisson (λ), then conditional on $\sum_{i=1}^{n} X_i = t$, $X_1 \sim \text{Bi}\left(t, \frac{1}{n}\right)$. So then in the iid Poisson (λ) model, consider estimation of

$$\gamma(\lambda) = (1+\lambda)\exp(-\lambda) = P_\lambda\left[X_1 = 0 \text{ or } 1\right]$$

a) Identify the UMVUE of this quantity and argue carefully that your estimator is indeed UMVU.

b) What is the Cramér-Rao lower bound for the variance of your estimator from a)? Do you expect that this bound is achieved? Why or why not?

2. Suppose that a discrete random variable X has pmf $f(x|\theta)$ specified in the table below for some θ.

		x = 1	2	3	4	5
θ	4	5/15	4/15	3/15	2/15	1/15
	3	4/30	5/30	6/30	7/30	8/30
	2	1/15	2/15	3/15	4/15	5/15
	1	.2	.2	.2	.2	.2

a) Identify a test $\phi(x)$ that is MP size $\alpha = .3$ for testing $\mathrm{H}_0\!:\theta = 1$ vs $\mathrm{H}_a\!:\theta = 2$ and carefully state why it has this property.

b) Your test from a) could be used to test $\mathrm{H}_0\!:\theta = 1$ vs $\mathrm{H}_a\!:\theta = 2 \text{ or } 3$. In this context, is it UMP of its size? Explain carefully.

c) Either identify a UMP size $\alpha = .4$ test of $\mathrm{H}_0\!:\theta = 1$ vs $\mathrm{H}_a\!:\theta = 2 \text{ or } 4$ or argue very carefully that no such test exists.

3. Let X be exponential with mean β, i.e. have pdf $f(x|\beta) = I[x>0]\frac{1}{\beta}\exp\left(-\frac{x}{\beta}\right)$ on $\Re$. Let

$$Y = X \text{ rounded to the nearest integer}$$

be a "digitized" version of X, so that Y has pmf

$$h(y|\beta) = \begin{cases} 1-\exp\left(-\frac{.5}{\beta}\right) & y=0 \\ \exp\left(-\frac{y-.5}{\beta}\right)-\exp\left(-\frac{y+.5}{\beta}\right) & y=1,2,3,\ldots \\ 0 & \text{otherwise} \end{cases}$$

It is possible to show (don't try to do so here) that for $\beta_1 < \beta_2$

$$\frac{h(0|\beta_2)}{h(0|\beta_1)} < \frac{h(1|\beta_2)}{h(1|\beta_1)}$$

a) Show that the family of distributions specified by $h(y|\beta)$ has MLR in y.

b) Identify a UMP test of size $\alpha = .2212$ of $\mathrm{H}_0{:}\beta \geq 10$ vs $\mathrm{H}_\mathrm{a}{:}\beta < 10$ based on the digitized observation Y.

4. A curious statistician is interested in the properties of a 5-dimensional distribution on $[0,1]^5$ that has pdf proportional to

$$p(\mathbf{x}) = \exp\left[-\left((x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_4)^2 + (x_4 - x_5)^2 + (x_5 - x_1)^2\right)\right]$$

This person decides to simulate from this distribution.

Carefully describe EITHER a rejection algorithm that could be used to produce realizations $\mathbf{x}$ from this distribution OR a "Gibbs Sampler"/"Successive Substitution Sampling" algorithm that could be used to draw samples from this distribution.

(If you choose to describe a Gibbs sampler, give a formula for the univariate pdf that must be sampled when replacing the *i*th entry of a current $\mathbf{x}$ vector. This formula will involve a 1-dimensional integral that would have to be evaluated numerically.)

5. Suppose that X has pdf $f\left(x\mid\theta\right)=2\theta\left(1-2x\right)+2x$ on $\left[0,1\right]$ for $\theta\in\Theta=\left[0,1\right]$. A Bayesian wants to test $\mathrm{H}_0\!:\theta\le .4$ vs $\mathrm{H}_\mathrm{a}\!:\theta>.4$. If the Bayesian's prior distribution is uniform on $\left[0,1\right]$, what is the person's (0-1 loss optimal) test?

6. Suppose that X is discrete with pmf $f\left(x\mid\theta\right)$ and that $T\left(X\right)$ is sufficient for $\theta\in\Theta\subset\Re$. Let $g\left(t\mid\theta\right)$ be the pmf for $T\left(X\right)$ and $h\left(x\mid t\right)$ be the conditional pmf for X given that $T\left(X\right)=t$. We may then write

$$f\left(x\mid\theta\right)=h\left(x\mid T\left(x\right)\right)g\left(T\left(x\right)\mid\theta\right)$$

Suppose that all useful regularity holds (for example, that $g\left(t\mid\theta\right)$ is differentiable in θ for all t). Argue very carefully that the Fisher information in $T\left(X\right)$ about θ at θ_0 is the same as the Fisher information in X about θ at θ_0 (i.e. that $I_{T(X)}\left(\theta_0\right)=I_X\left(\theta_0\right)$).