Stat 543 Exam 3
Spring 2016

I have neither given nor received unauthorized assistance on this exam.

KEY

Name Signed Date

Name Printed

This exam consists of 11 questions. Do at least 9 of them. I will score answers at 10 points apiece and count you best 9 scores (making 90 points possible).

1. A particular continuous model for random pairs (X,Y) with parameter $\gamma\in(0,\infty)$ has joint pdf

$$f(x,y\mid\gamma)=\frac{\gamma}{2\pi\sqrt{x^2+y^2}}\exp\left(-\gamma\sqrt{x^2+y^2}\right)\quad\forall(x,y)\in\Re^2$$

(This "radially symmetric" density is constant on circles centered at the origin.)

a) For n iid data pairs $(X_1,Y_1),\ldots,(X_n,Y_n)$ identify a statistic in which there is monotone likelihood ratio (and show that your statistic really does the job).

Let $R_i=\sqrt{X_i^2+Y_i^2}$. The likelihood is $\prod_{i=1}^n f(x_i,y_i\mid\gamma)$

i.e. the likelihood is $\frac{\gamma^n}{(2\pi)^n\prod_{i=1}^n r_i}\exp-\gamma(\Sigma r_i)$. So for

$\gamma_2>\gamma_1$ the likelihood ratio is

$$\Lambda(\gamma_2,\gamma_1)=\left(\frac{\gamma_2}{\gamma_1}\right)^n\exp\left(\left[-\sum_{i=1}^n r_i\right](\gamma_2-\gamma_1)\right)$$

which is monotone increasing in $\left(-\sum_{i=1}^n r_i\right)$

b) For the $n=1$ case, find the UMP size $\alpha=.05$ test of $\mathrm{H}_0:\gamma\le 1$ vs $\mathrm{H}_a:\gamma>1$ in as explicit a form as is possible. (You may recall that change of variables to polar coordinates in a double integral involves replacing $dxdy$ with $rdrd\theta$.)

We want to reject H_0 for large values of $T=-R$ i.e. small values of R. The cut-off value c needs to be chosen so that $P_{\gamma=1}[R<c]=.05$

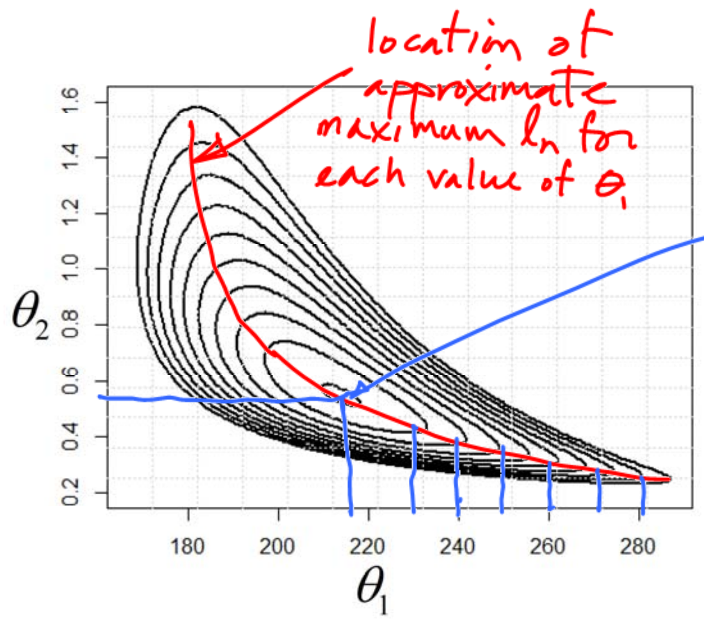
$$P_{\gamma=1}[R<c]=\iint f(x,y\mid 1)\,dx\,dy$$

$$=\int_0^{2\pi}\int_0^c\frac{1}{2\pi r}\exp(-r)\,r\,dr\,d\theta=1-e^{-c}$$

So we want $.05=1-e^{-c}$ i.e. $e^{-c}=.95$ i.e.

$c=-\ln(.95)$. The test is $\phi(x,y)=I\left[\sqrt{x^2+y^2}<-\ln(.95)\right]$.

2. Below is a cartoon of a contour plot for a loglikelihood function $l_n(\theta_1,\theta_2)$. Find approximate values for a maximum likelihood estimate of $\boldsymbol{\theta}=(\theta_1,\theta_2)$ and the values of the profile loglikelihood for θ_1, $l_n^*(\theta_1)$. (The tight contours are at large values of the logliklihood. Presume that the tightest contour is at -10 and that they fall off at 5 units per contour, so that they are at $-10,-15,-20,\dots$.)



$\hat{\boldsymbol{\theta}}^{\text{MLE}} \approx$ (215, .55)

Contour loglikelihood values are:

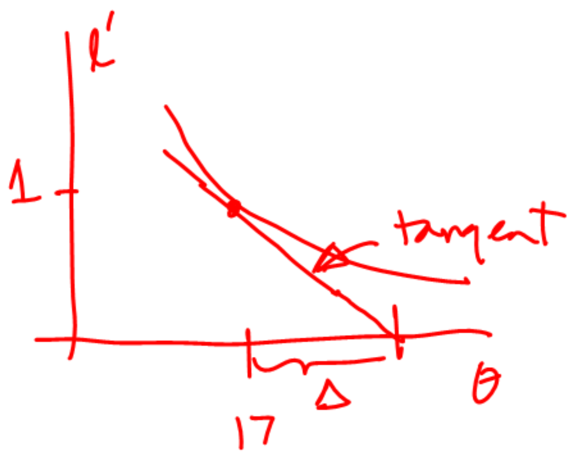
$l_n^*(230) \approx$ −14.5

$l_n^*(240) \approx$ −19.5

$l_n^*(250) \approx$ −25.0

$l_n^*(260) \approx$ −34 $\quad$ $l_n^*(270) \approx$ −43 $\quad$ $l_n^*(280) \approx$ −50

3. A method of moments estimate of a parameter θ is $\hat{\theta}=17$. The loglikelihood function, $l(\theta)$, is complicated, but it is possible to find (numerical) first and second derivatives $l'(17)=1$ and $l''(17)=$ $-.2$.
Find a correction/improvement on the method of moments estimate based on these derivatives.



$l'(\theta)=0$ has a root that can be approximated as

$17+\Delta$ where $-\frac{1}{\Delta}=l''(17)$

i.e. $\Delta=\frac{1}{.2}=5$

So the "improved" estimate is $17+5=22$.

4. A loglikelihood for a parameter vector $\boldsymbol{\theta} = (\theta_1, \theta_2)$ is approximately quadratic near $\hat{\boldsymbol{\theta}}^{\text{MLE}} = \left(\frac{1}{2}, \frac{1}{2}\right)$,

$$l_n(\theta_1, \theta_2) \approx -2\theta_1^2 + 2\theta_1\theta_2 - \theta_2^2 + \theta_1$$

Based on a large sample approximation to the distribution of a vector MLE, use the Wald method to give an approximately 95% two-sided confidence interval for θ_1.

2nd partials of l_n are $\frac{\partial^2}{\partial\theta_1^2} l_n = \frac{\partial}{\partial\theta_1}(-4\theta_1 + 2\theta_2 + 1) = -4$
$\frac{\partial^2}{\partial\theta_2^2} l_n = \frac{\partial}{\partial\theta_2}(2\theta_1 - 2\theta_2) = -2$ and $\frac{\partial^2}{\partial\theta_1\partial\theta_2} l_n = 2$. So the inverse negative Hessian at $(\frac{1}{2}, \frac{1}{2})$ is $\begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -2 \\ -2 & 4 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$

So an estimated variance of the 1st co-ordinate of $\hat{\underset{\sim}{\theta}}^{\text{MLE}}$ is $\frac{1}{2}$. So an approximate 95% interval is

$$\hat{\theta}_1^{\text{MLE}} \pm z\sqrt{\text{est'd variance}} \quad \text{i.e.} \quad \frac{1}{2} \pm 1.96\sqrt{\frac{1}{2}}$$

5. Suppose that $k(\lambda)$ is the smallest integer k so that for $X \sim \text{Poisson}(\lambda)$, $P[X \geq k] \leq .05$. You may think of plotting $k(\lambda)$ and producing an increasing step function taking integer values. If I observe a Poisson variable X, exactly how should I use that step function to produce a confidence interval for λ with associated confidence at least 95%?

We can invert a set of size $\alpha \leq .05$ tests of $H_0: \lambda = \lambda_0$. Test for λ_0 is $I[X \geq k(\lambda_0)]$. Then the set of λ_0 for which H_0 is not rejected serves as a confidence set. This is the set of λ_0 for which $X < k(\lambda_0)$. This is the set of λ_0 for which the step function takes the value $X+1$ or larger.

6. For parameters $0 < c < 1$ and $0 < p < 1$, a pdf on $(0,1)$ is of the form

$$f(x \mid c, p) = \begin{cases} \dfrac{p}{c} & \text{for } 0 < x < c \\ \dfrac{1-p}{1-c} & \text{for } c < x < 1 \end{cases}$$

(The density is constant to the left and then to the right of c, assigning probabilities $p = P[X \le c]$ and $1 - p = P[X > c]$.) Find the form of the likelihood ratio statistic for testing $\mathrm{H}_0 : c = .5$ based on n iid observations from this density, $X_1, \ldots, X_n$.

For any c, let $n(c) = \#[X_i \le c]$

$$\lambda(\underset{\sim}{x}) = \frac{\max_{p,c} \prod_{i=1}^{n} f(x_i \mid c, p)}{\max_{p} \prod_{i=1}^{n} f(x_i \mid .5, p)}$$

For any given c, $\prod_{i=1}^{n} f(x_i \mid c, p) \propto p^{n(c)} (1-p)^{n-n(c)}$ which is maximized at $\hat{p}(c) \equiv \frac{n(c)}{n}$. So define

$$\hat{m}(c) = \left(\frac{\hat{p}(c)}{c}\right)^{n(c)} \left(\frac{1-\hat{p}(c)}{1-c}\right)^{n-n(c)}$$

The denominator of λ is $\hat{m}(\frac{1}{2})$. The numerator is $\max_c \hat{m}(c)$. Note that $n(c)$ and thus $n - n(c)$, $p(c)$, and $1-p(c)$ change only at X order statistics. For c between order statistics $c^{n(c)}(1-c)^{n-n(c)}$ is minimized at one of the order statistics. So $\hat{m}(c)$ is maximized by searching over $c = X_{(1)}, X_{(2)}, \ldots, X_{(n)}$. This amounts to examining the n values

$$\left(\frac{j/n}{X_{(j)}}\right)^{j} \left(\frac{1 - j/n}{1 - X_{(j)}}\right)^{n-j}$$

and choosing the j maximizing this, say j^*. Then

$$\lambda(\underset{\sim}{x}) = \hat{m}\left(\tfrac{j^*}{n}\right) / \hat{m}\left(\tfrac{1}{2}\right)$$

7. A discrete distribution has pmf

$$f(x\,|\,p) = \begin{cases} \frac{1}{2}\left(2p - p^2\right) & x = 1,2 \\ p(1-p)^{x-1} & x = 3,4\ldots \end{cases}$$

Find a 2-dimensional sufficient statistic based on a sample $X_1, X_2, \ldots, X_n$ iid from this distribution, and say carefully how you know that it is sufficient. Let $n^* = \#[x_i = 1,2]$

Then $\prod_{i=1}^{n} f(x_i|p) = \left(\frac{1}{2}(2p-p^2)\right)^{n^*} p^{n-n^*} (1-p)^{\sum_{x_i>2} x_i - n^*}$

which with $g(p,m,t) = \left(\frac{1}{2}(2p-p^2)\right)^m p^{n-m} (1-p)^{t-m}$

is $g(p, n^*, \sum_{x_i>2} x_i)$ so that by the factorization theorem

$(n^*, \sum_{X_i>2} X_i)$ is sufficient for p

8. For a distribution on $\{1,2,\ldots,M\}$ specified by values $p_1, p_2, \ldots, p_M$, the quantity

$$\mathcal{E}(\mathbf{p}) = -\sum_{m=1}^{M} p_m \ln(p_m)$$

is called the "entropy" of the distribution. Using a statistical "information" measure, show that this is no larger than $\ln(M)$ (the entropy of the uniform distribution). (In the above expression the convention that $0 \cdot \ln(0) = 0$ is in force.)

$$\mathcal{E}(\underset{\sim}{p}) = -\sum_{m=1}^{M} p_m \ln(p_m) - \sum_{m=1}^{M} p_m \ln(M) + \ln(M)$$

$$= -\left(\sum_{m=1}^{M} p_m \frac{\ln(p_m)}{\ln(\frac{1}{M})}\right) + \ln(M)$$

$$= -K(\underset{\sim}{p}, \underset{\sim}{u}) + \ln(M) \leq \ln(M)$$

K-L information is ≥ 0

uniform dsn on $\{1,2,\ldots,M\}$

9. A pmf on the integers with an integer parameter, θ, is

$$f(x\,|\,\theta)=\frac{1}{2}I\left[x=\theta-1 \text{ or } x=\theta+1\right]$$

For X_1 and X_2 iid from this distribution, compare MSE's for the two estimators of θ,

$$\hat{\theta}=\begin{cases} X_1+1 & \text{if } X_1=X_2 \\ \bar{X} & \text{if } X_1\neq X_2 \end{cases} \quad \text{and} \quad \tilde{\theta}=\hat{\theta}-\frac{1}{2}$$

10. Suppose that $X\sim \mathrm{U}\left(\theta-\frac{1}{2},\theta+\frac{1}{2}\right)$ and that a prior distribution for θ is $\mathrm{N}(0,1)$. Find the Bayes estimator of θ under squared error loss.