Stat 543 Exam 3
Spring 2016

I have neither given nor received unauthorized assistance on this exam.

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Name Signed **Date**

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Name Printed

This exam consists of 11 parts. Do at least 8 of them. I will score answers at 10 points apiece and count you best 8 scores (making 80 points possible).

1. A particular continuous model for random pairs (X,Y) with parameter $\gamma \in (0,\infty)$ has joint pdf

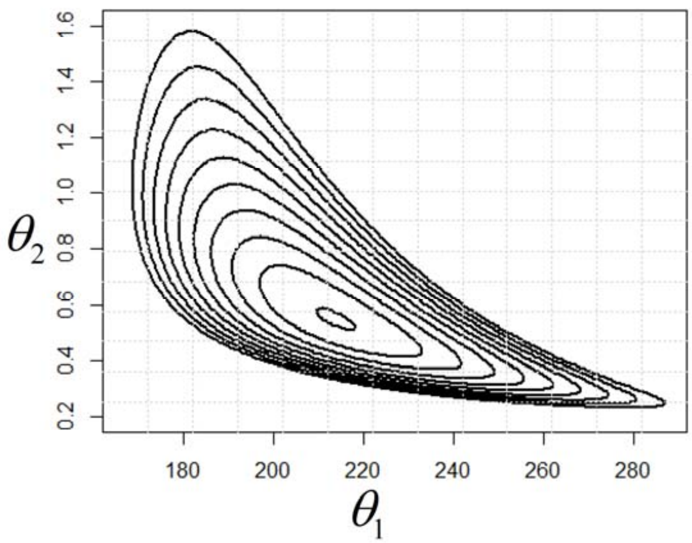
$$f(x,y \mid \gamma) = \frac{\gamma}{2\pi\sqrt{x^2+y^2}} \exp\left(-\gamma\sqrt{x^2+y^2}\right) \quad \forall (x,y) \in \Re^2$$

(This "radially symmetric" density is constant on circles centered at the origin.)

a) For n iid data pairs $(X_1,Y_1),\ldots,(X_n,Y_n)$ identify a statistic in which there is monotone likelihood ratio (and show that your statistic really does the job).

b) For the $n=1$ case, find the UMP size $\alpha = .05$ test of $\mathrm{H}_0 : \gamma \le 1$ vs $\mathrm{H}_a : \gamma > 1$ in as explicit a form as is possible. (You may recall that change of variables to polar coordinates in a double integral involves replacing $dxdy$ with $rdrd\theta$.)

2. Below is a cartoon of a contour plot for a loglikelihood function $l_n(\theta_1,\theta_2)$. Find approximate values for a maximum likelihood estimate of $\boldsymbol{\theta}=(\theta_1,\theta_2)$ and the values of the profile loglikelihood for θ_1, $l_n^*(\theta_1)=\max_{\theta_2} l_n(\theta_1,\theta_2)$. (The tight contours are at large values of the loglikelihood. Presume that the tightest contour is at -10 and that they fall off at 5 units per contour, so that they are at $-10,-15,-20,\ldots$.)



$\hat{\boldsymbol{\theta}}^{\text{MLE}} \approx$ ________________

Profile loglikelihood values are:

$l_n^*(230) \approx$ __________

$l_n^*(240) \approx$ __________

$l_n^*(250) \approx$ __________

$l_n^*(260) \approx$ __________

$l_n^*(270) \approx$ __________ $l_n^*(280) \approx$ __________

3. A method of moments estimate of a parameter θ is $\hat{\theta}=17$. The loglikelihood function, $l(\theta)$, is complicated, but it is possible to find (numerical) first and second derivatives $l'(17)=1$ and $l''(17)=-.2$. Find a correction/improvement on the method of moments estimate based on these derivatives.

4. A loglikelihood for a parameter vector $\boldsymbol{\theta} = (\theta_1, \theta_2)$ is approximately quadratic near $\hat{\boldsymbol{\theta}}^{\text{MLE}} = \left(\frac{1}{2}, \frac{1}{2}\right)$,

$$l_n(\theta_1, \theta_2) \approx -2\theta_1^2 + 2\theta_1\theta_2 - \theta_2^2 + \theta_1$$

Based on a large sample approximation to the distribution of a vector MLE, use the Wald method to give an approximately 95% two-sided confidence interval for θ_1.

5. Suppose that $k(\lambda)$ is the smallest integer k so that for $X \sim \text{Poisson}(\lambda)$, $P[X \geq k] \leq .05$. You may think of plotting $k(\lambda)$ and producing an increasing step function taking integer values. If I observe a Poisson variable X, exactly how should I use that step function to produce a confidence interval for λ with associated confidence at least 95%?

6. For parameters $0 < c < 1$ and $0 < p < 1$, a pdf on $(0,1)$ is of the form

$$f(x \mid c, p) = \begin{cases} \dfrac{p}{c} & \text{for } 0 < x < c \\ \dfrac{1-p}{1-c} & \text{for } c < x < 1 \end{cases}$$

(The density is constant to the left and then to the right of c, assigning probabilities $p = P[X \le c]$ and $1 - p = P[X > c]$.) Find the likelihood ratio statistic for testing $\mathrm{H}_0 : c = .5$ based on n iid observations from this density, $X_1, \ldots, X_n$ in as explicit a form as is possible.

7. A discrete distribution on the positive integers has pmf

$$f(x \mid p) = \begin{cases} \frac{1}{2}\left(2p - p^2\right) & x = 1, 2 \\ p(1-p)^{x-1} & x = 3, 4, \ldots \end{cases}$$

Find a 2-dimensional sufficient statistic based on a sample $X_1, X_2, \ldots, X_n$ iid from this distribution, and say carefully how you know that it is sufficient.

8. For a distribution on $\{1, 2, \ldots, M\}$ specified by values $p_1, p_2, \ldots, p_M$, the quantity

$$\mathcal{E}(\mathbf{p}) = -\sum_{m=1}^{M} p_m \ln(p_m)$$

is called the "entropy" of the distribution. Using a statistical "information" measure, show that this is no larger than $\ln(M)$ (the entropy of the uniform distribution). (In the above expression the convention that $0 \cdot \ln(0) = 0$ is in force.)

9. A pmf on the integers with an integer parameter, θ, is

$$f(x \mid \theta) = \frac{1}{2} I\left[x = \theta - 1 \text{ or } x = \theta + 1\right]$$

For X_1 and X_2 iid from this distribution, compare MSE's for the two estimators of θ,

$$\hat{\theta} = \begin{cases} X_1 + 1 & \text{if } X_1 = X_2 \\ \bar{X} & \text{if } X_1 \neq X_2 \end{cases} \quad \text{and} \quad \tilde{\theta} = \hat{\theta} - \frac{1}{2}$$

10. Suppose that $X \sim \mathrm{U}\left(\theta - \frac{1}{2}, \theta + \frac{1}{2}\right)$ and that a prior distribution for θ is $\mathrm{N}(0,1)$. Find the Bayes estimator of θ under squared error loss.